ROUGH SIMILARITY OF LEFT-INVARIANT RIEMANNIAN METRICS ON SOME LIE GROUPS

ENRICO LE DONNE, GABRIEL PALLIER, AND XIANGDONG XIE

Abstract. We consider Lie groups that are either Heintze groups or Sol-type groups, which generalize the three-dimensional Lie group SOL. We prove that all left-invariant Riemannian metrics on each such a Lie group are roughly similar via the identity map. This allows us to reformulate in a common framework former results by Le Donne-Xie, Eskin-Fisher-Whyte, Carrasco Piaggio, and recent results of Ferragut and Kleiner-Müller-Xie, on quasiisometries of these solvable groups.

Contents

1. Introduction

1.1. **Main results.** In this paper, we compare left-invariant Riemannian metrics on certain simply connected solvable Lie groups. The groups under study fall within two classes:

- Heintze groups, that is, simply connected solvable groups with Lie algebra $\mathfrak{s}$ such that $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ has codimension 1 in $\mathfrak{s}$ and $\mathfrak{s}$ splits as $\mathfrak{n} \rtimes \mathbf{R}$, where $\mathbf{R}$ acts on $\mathfrak{n}$ via a derivation D whose eigenvalues have positive real parts.
- Sol-type groups, that is, simply connected solvable groups with Lie algebra $\mathfrak{g}$ such that $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ has codimension 1 in $\mathfrak{g}$ and $\mathfrak{g}$ splits as $\mathfrak{n} \rtimes \mathbf{R}$, where $\mathbf{R}$ acts on $\mathfrak{n}$ via a derivation D whose eigenvalues have nonzero real parts, not all of the same sign, and such that

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$[\mathfrak{n}^{>0}, \mathfrak{n}^{<0}] = 0$, where $\mathfrak{n}^{<0}$ (resp. $\mathfrak{n}^{>0}$) is the sum of generalized eigenspaces with negative (resp. positive) real part.

Some of the relevant properties of these groups will be recalled in §2; the main common feature of the groups we consider is to be simply connected, solvable, and have one-dimensional first cohomology, though the latter do not constitute a characterization.

In order to state our main result, recall that if $\phi\colon X \to Y$ is assumed to be a quasiisometry between metric spaces X and Y, then for some $c \geqslant 0$ there are positive constants λ_- and λ_+ such that $\lambda_- d(x,x') - c \leqslant d(\phi(x),\phi(x'))$ and $\lambda_+ d(x,x') + c \geqslant d(\phi(x),\phi(x'))$ for every $x, x' \in X$. Say that the quasiisometry ϕ is a *rough similarity* if one can further take $\lambda_- = \lambda_+$ and a *rough isometry* if one can take $\lambda_- = \lambda_+ = 1$ in the inequalities above.

Theorem A. *Let S be a Heintze group and let g_1 and g_2 be left-invariant Riemannian metrics on S with distance function d_1 and d_2, respectively. Then the identity map $Id : (S, d_1) \to (S, d_2)$ is a rough similarity.*

Theorem B. *Let G be a Sol-type group and g_1 and g_2 be left-invariant Riemannian metrics on G with distance function d_1 and d_2, respectively. Then the identity map $Id : (G, d_1) \to (G, d_2)$ is a rough similarity.*

Theorems A and B imply the following statement at no cost: if φ is an automorphism of a Heintze or Sol-type group, then φ is a rough similarity with respect to every left-invariant Riemannian metric. (It is an elementary fact that the *inner* automorphisms of any group G equipped with a left-invariant distance d are rough isometries; however, the continuous group automorphisms are in general no more than quasiisometries assuming in addition that G is compactly generated, that d is proper geodesic.)

Using Theorems A and B we are able to reformulate certain results that appeared separately in the literature.

In the statement below, a Heintze group is of *special type* if it is a closed co-compact subgroup of a rank-one simple Lie group; *Carnot type* is a subclass of Heintze groups in which the nilradical is a Carnot group, and the derivation D is a Carnot derivation of this group. For the background on Carnot groups, see for example [LD17]. The real shadow construction will be recalled along with precise definitions in §2.3.1.

The substantial part of the following theorem is provided by the given references, while its formulation depends on the results above.

Theorem C. *Let G belong to the following list:*

1. *The Lie group* SOL [EFW13].
2. *Heintze group whose real shadow is of Carnot type with reducible first stratum* [LDX16].
3. *Heintze group whose real shadow is not of Carnot type* [CP17].
4. (a) *Heintze group whose real shadow is of Carnot type, which is different from the special-type subgroups in* $\mathrm{SO}(n,1)$ *or* $\mathrm{SU}(n,1)$*, and whose nilradical is nonrigid in the sense of Ottazzi-Warhurst* [KMX21].
   (b) *The Carnot-type Heintze group over the subgroup of unipotent triangular real $n \times n$ matrices, $n \geqslant 4$* [KMX22].
5. *Non-unimodular Sol-type group* [Fer22].

Equip G with any left-invariant Riemannian metric with associated distance d. If $\phi\colon G \to G$ is a quasiisometry, then ϕ is a rough isometry with respect to d.

Note that, in general, the notion of a rough isometry of a group does not make sense because it depends on the left-invariant distance one choses on the group. In view of Theorems A and B, the conclusion of Theorem C may also be stated in the following way: given any pair of left-invariant Riemannian distances d_1 and d_2, every quasiisometry $(G, d_1) \to (G, d_2)$ is a rough similarity, whose similarity constant only depends on the pair (d_1, d_2).

We point out that the rigidity property of quasiisometries expressed in Theorem C is weaker than the rigidity of quasiisometries (which means every self quasiisometry of a certain metric space is at a finite distance from an isometry). Every map at a finite distance from a isometry is a rough isometry. However, depending on the space there may exist rough isometries that are not at finite distance from any isometry, and this does actually happen for the left-invariant metrics on certain Heintze and Sol-type groups[1].

We also note the following:

- Carrasco Piaggio has stated the conclusion in an equivalent form when G is as in (3) and additionally purely real [CP17]. His result subsumes former ones, the first of which being by Xie and Shanmugalingam [SX12], the second one by Xie in [Xie14].
- Case (2) subsumes former work by Xie in [Xie13]. The groups of class (C) defined in [Pan89b, 14.1] fall within this family (See Remark 5.2), and the early [Pan89b, Theorem 4] implies Theorem C for these: their quasiisometries are actually a bounded distance away from inner automorphisms.
- Cases (2) and (4a) overlap, though none of them imply the other. The groups considered in [Pan89b, §14.3] belong to both classes. Case (4b) is not implied by (2) nor by (4a).

Bringing them together, the cases (2), (3) and (4) of Theorem C support the following conjecture:

Conjecture D. Let S be a Heintze group, that is not among the special-type subgroups of $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ for any $n \geqslant 2$. Equip S with any left-invariant Riemannian metric. Then every self-quasiisometry of S is a rough isometry.

We will discuss further the relations and differences of Theorem C and Conjecture D with quasi-isometric rigidity in the case of Heintze groups in §1.2.2. Especially, we will see there that Conjecture D would follow from conjectures already explicitly stated in [KMX21] and [Cor18]. Keeping in mind that every homogeneous space of negative curvature is a Heintze group with a left-invariant metric, Conjecture D can be considered as a precise version of the feeling expressed in the four lines before §1 in [Pan89b].

1.2. **Some context.**

1.2.1. *Spaces of left-invariant metrics and comments on Theorems A and B.* The space of left-invariant Riemannian metrics on a given Lie group has been widely studied by differential geometers; let us rather restrict our discussion to the results that put an emphasis on large-scale geometry rather than on Lie groups, for we believe that this comparison is more instructive.

For a finitely generated group Γ, Gromov introduced a metric space denoted by WM_Γ whose points are word metrics and the distance is measured by the logarithm of λ, where $(1/\lambda, \lambda)$ is the

[1] See e.g. [Pan89b, 14.3] for such Heintze groups. Considering the three-dimensional group SOL with Riemannian metric $ds^2 = dt^2 + e^{-2t}dx^2 + e^{2t}dy^2$, any self-map of the form $(x, y, t) \mapsto (x, f(y), t)$, where f is a bilipschitz homeomorphism of $\mathbb{R}$ which is not affine, is a rough isometry not a bounded distance away from an isometry.

optimal pair of multiplicative quasiisometry constant between them [Gro93]. The definition of this space itself is not straightforward, as one may consider several variants, especially one could compare metrics only through the identity map (as we do here), or through automorphisms, or even through arbitrary maps[2]. One may also include metrics that are not word metrics, especially geometric metrics, induced by the Riemannian metrics on universal covers when Γ is the fundamental group of a compact manifold. The resulting space is in some sort reminiscent of Teichmüller space, and actually contains it when Γ is a surface group.

Recently one of the variants of this space of left-invariant metrics was studied by Oregón-Reyes in the case of word hyperbolic groups [OR22, Theorem 1.3]. Oregón-Reyes notes the analogy with Teichmüller spaces and identifies metrics that are roughly similar through the identity.

Theorem (Oregón-Reyes)**.** *Let Γ be a word-hyperbolic group. Consider the space $\mathcal{D}(\Gamma)$ of left-invariant metrics on Γ that are quasiisometric to word metrics, modded out by the equivalence relation $d \sim d'$ if d and d' are roughly similar through the identity. Equip $\mathcal{D}(\Gamma)$ with the metric*

$$\rho(d,d') := \inf\{\log\lambda : \exists\sigma>0, \exists c\geqslant 0,\ \frac{\sigma}{\lambda}d - c \leqslant d' \leqslant \sigma\lambda d + c\}, \qquad \forall d,d'\in\mathcal{D}(\Gamma).$$

Then $\mathcal{D}(\Gamma)$ is unbounded.

All the Heintze groups being Gromov-hyperbolic, Oregón-Reyes result is in sharp contrast with ours, which suggests that Theorems A and B may be special to non-finitely generated groups. Whether they are special to connected Lie group is currently unknown to us and we ask specific questions in this direction at the end of this paper.

1.2.2. *Differences with other forms of rigidity.* Some of the papers cited in Theorem C were dedicated to proving quasiisometric rigidity, and they are known for this, so that it may be useful to point out the differences of the conclusion of Theorem C with quasiisometric rigidity itself. Namely, the following is expected:

QI Rigidity Conjecture. Let Γ be a finitely generated group.

(1) If Γ is quasiisometric to a Heintze group S, then S is of special type, and Γ is virtually a lattice in the rank-one simple Lie group containing S as a co-compact closed subgroup.
(2) If Γ is quasiisometric to a Sol-type group G, then G is unimodular, and Γ is virtually a lattice in a Lie group $\widehat{G}$ containing G as a co-compact closed subgroup.

A common significant ingredient between quasiisometric rigidity and Theorem C can be singled out in the case of Heintze groups that are not of special type. It is the following.

Pointed sphere Conjecture (Cornulier, [Cor18, 19.104])**.** Let ϕ be a self quasiisometry of a Heintze group, not of special type. Then the extension of ϕ to the Gromov boundary of S fixes the unique boundary point that is fixed by all left-translations of S.

While conjectural in general, the following scheme of proof for Conjecture D should help the reader to understand our approach of some of the special cases of it in the present paper.

Proof of Conjecture D assuming the Pointed Sphere Conjecture and [KMX20, Conjecture 1.13]. (See Figure 1). Let $S = N \rtimes \mathbb{R}$ be a Heintze group as in the statement of Conjecture D. Then,

[2]One should also decide if roughly isometric or roughly similar metrics are to be identified; however this is not a deep distinction.

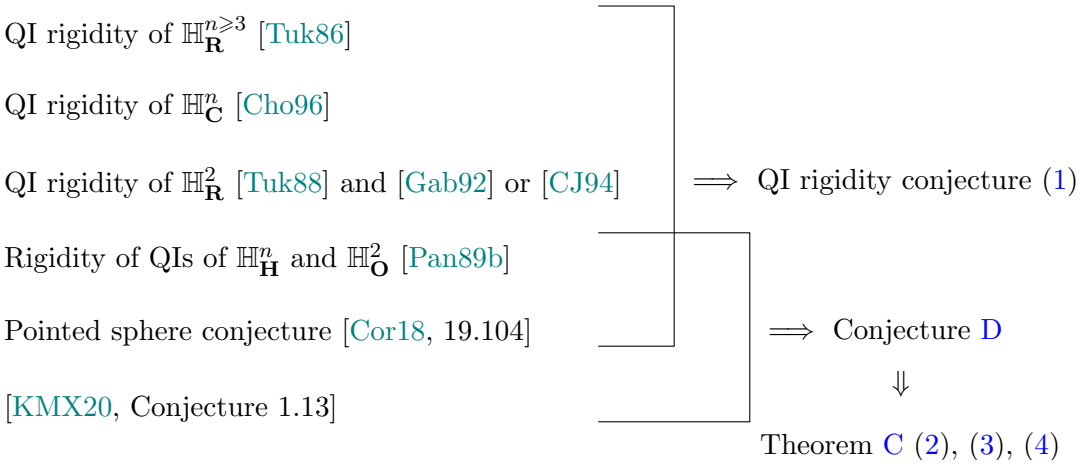

FIGURE 1. Common ingredients for the QI rigidity conjecture for Heintze groups and Conjecture D.

- Either S is not of special type. In this case, the Gromov boundary of S can be identified with a one-point compactification of N, with the boundary extension of ϕ stabilizing N. By [KMX20, Conjecture 1.13], then, the boundary extension of ϕ to N equipped with a Carnot-Carathéodory metric should be bilipschitz, which, by [SX12], implies that ϕ is a rough isometry.
- Or S is of special type. Then, by assumption, it is a closed cocompact subgroup of $\mathrm{Sp}(n,1)$ or $F_4^{(-20)}$ for some $n \geqslant 2$. The quasiisometries of S are at a bounded distance from isometries of a left-invariant symmetric Riemannian metric on S by [Pan89b], which implies by Theorem A that they are rough isometries of any left-invariant Riemannian metric, as mentioned in the paragraph below Theorem C. □

The QI rigidity conjecture for Heintze groups, on the other hand, would follow from a combination of the QI rigidity for special-type groups, which were obtained in the 1980s and early 1990s (See Figure 1), together with the fact that no finitely generated group should be quasi-isometric to a non-special Heintze group. We refer to [SX12, Proof of Corollary 1.3] for how the Pointed Sphere Conjecture implies the last statement.

Finally, an analogy coming from the world of finitely-generated groups may lead one to think of quasiisometries of groups as large-scale counterparts of homotopy equivalences between compact manifolds. Following this analogy, rough isometries of fundamental groups of negatively curved closed manifolds are large-scale counterparts to those homotopy equivalences that identify the marked length spectra [Fuj16]. Theorem C may then be considered analogous to the rigidity result that would consist in upgrading homotopy equivalence to marked length spectra isomorphism. Mostow's rigidity, which goes from homotopy equivalence to isometry, is strictly stronger, while length spectrum rigidity, which goes from the length spectrum to the isometry type, measures the difference.

1.3. **Organization of the paper.** Section 2 collects preliminary material, namely definitions and three lemmas from Gromov-hyperbolic geometry. Section 3 proves Theorem A and Section 4

proves Theorem B. Section 4 is the technical heart of the paper, and Theorem B is significantly harder to prove than Theorem A. In Section 5 we start by proving a special case of Cornulier's Pointed Sphere Conjecture, which is instrumental in the reformulation of the main theorem of [LDX16]. Next, we prove the other cases of Theorem C. In Section 6 we point out that the conclusion of Theorem B does not hold for the Lamplighter group, and suggest a strengthening of the conclusion expressed by Theorems A and B which would be formulated in term of geometric actions that we did not reach in this paper.

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2. Preliminary

2.1. **Notation.** If G, H, N, S are Lie groups then $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}, \mathfrak{s}$ are their Lie algebras.

2.2. **Gromov-hyperbolic geometry.** Let T be a tree, $\xi \in \partial T$ a point in the ideal boundary, and $x, y \in T$. Then the intersection of the two rays $x\xi$ and $y\xi$ is also a ray: $x\xi \cap y\xi = z\xi$, where $x\xi$, $y\xi$ branch off at z. The distance $d(x, y)$ equals the distance from x to the branch point z plus the distance from y to the branch point z. A similar statement holds for all Gromov-hyperbolic spaces.

The following lemma follows easily from the thin triangle condition. We omit the proof.

Lemma 2.1 (See Figure 2)**.** *Let X be a proper geodesic δ-hyperbolic space, $\xi \in \partial X$, and $x, y \in X$. Then there is a constant C depending only on δ, points $x' \in x\xi$, $y' \in y\xi$ such that $d(x', y') \leq C$ and the concatenation $xx' \cup x'y' \cup y'y$ is a $(1, C)$-quasi-geodesic. Here $x\xi$ denotes any geodesic joining x and ξ; similarly for $y\xi$, xx', $x'y'$, $y'y$. In particular, $|d(x, y) - (d(x, x') + d(y, y'))| \leq C$. Furthermore, x', y' can be chosen so that they lie on the same horosphere centered at ξ.*

The next two lemmas are more involved, and will not be used before Section 4 where they serve as a preparation for the key step of Theorem B. The starting point is a well-known fact about simply connected Riemannian manifolds with sectional curvature bounded above by a negative constant: if p and q lie on the same horosphere, then the length of every path joining p and q outside the horoball is at least exponential in $d(p, q)$. For completeness, we provide a proof that also applies to Gromov-hyperbolic spaces.

Lemma 2.2. *Let X be a proper geodesic δ-hyperbolic space, $\xi \in \partial X$, S a horosphere centered at ξ, and B the horoball bounded by S. Then for every $p, q \in S$ and every path c in $X \backslash B$ joining p and q, the length of c satisfies $\ell(c) \geq 2^{\frac{d(p,q)-C-2}{2\delta}} - C$, where C is a constant depending only on δ.*

Proof. Let γ be a geodesic between p and q and r be a "highest" point on γ, that is, for any Busemann function b centered at ξ, we have $b(r) = \min\{b(x) | x \in \gamma\}$, see Figure 3. We claim $B(r, d(p, q)/2 - 2C_2) \subset B$ for some constant C_2 depending only on δ. To see this, we first notice that $d(r, p) \geq d(p, q)/2$ or $d(r, q) \geq d(p, q)/2$. Without loss of generality we assume $d(r, p) \geq d(p, q)/2$. Next we consider the path $\gamma[p, r] \cup r\xi$, where $\gamma[p, r]$ denotes the segment of γ between p and r. Since r is a "highest" point on γ, it is clear that $\gamma[p, r] \cup r\xi$ is a $(1, C_1)$

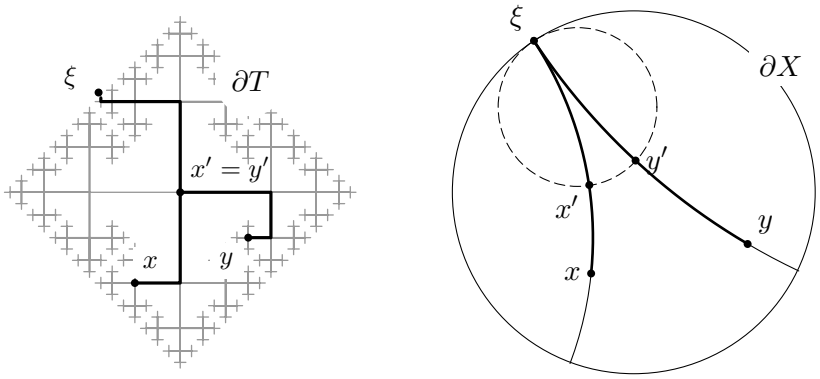

FIGURE 2. Lemma 2.1 in a tree and in the hyperbolic plane.

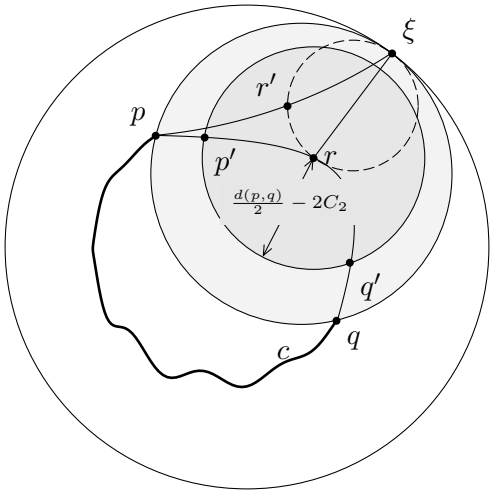

FIGURE 3. Proof of Lemma 2.2.

quasi-geodesic from p to ξ for some constant C_1 depending only on δ. By the Morse Lemma[3], the Hausdorff distance between $p\xi$ and $\gamma[p,r]\cup r\xi$ is bounded above by a constant C_2 depending only on δ. Hence $d(r,x) \le C_2$ for some $x \in p\xi$. Let $r' \in p\xi$ be the point at the same height as r, that is, $b(r') = b(r)$. Then $d(x,r') \le C_2$ (comparing the Busemann function of x and r' with respect to ξ) and so by the triangle inequality $d(r,r') \le 2C_2$. It follows that

$$b(r) - b(p) = d(r',p) \ge d(p,r) - d(r',r) \ge d(p,r) - 2C_2 \ge d(p,q)/2 - 2C_2.$$

The claim follows from this.

Let $p' \in \gamma$ between p and r such that $d(r,p') = d(p,q)/2 - 2C_2$, and $q' \in \gamma$ between r and q such that $d(r,q') = d(p,q)/2 - 2C_2$, see Figure 3. Then the path $c' = \gamma[p',p]\cup c\cup\gamma[q,q']$ joins p' and q' and lies outside the ball $B(r, d(p,q)/2 - 2C_2)$. By Proposition 1.6 on page 400 of [BH99], the length of c' satisfies

$$\ell(c') \ge 2^{\frac{d(p,q)/2 - 2C_2 - 1}{\delta}}.$$

[3]Incidentally, the version of the current lemma where γ avoids a ball rather than a horoball is a key ingredient in the proof of the Morse Lemma itself. So it actually occurs twice in this proof.

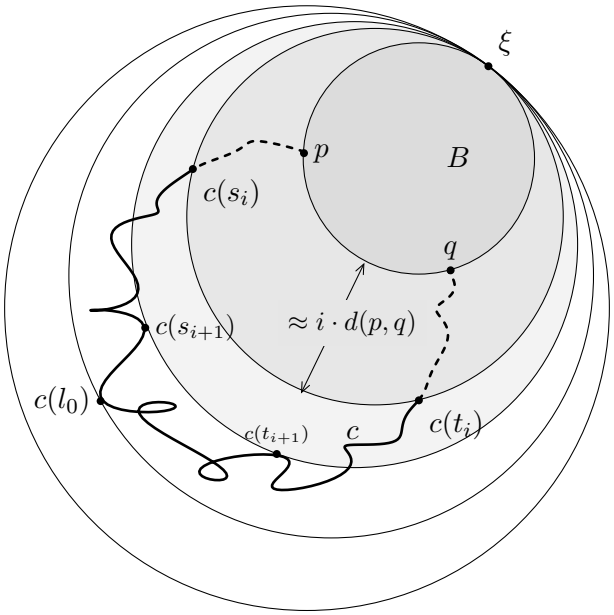

FIGURE 4. Proof of Lemma 2.3 in the situation where $H(c)$ is much larger than $d(p,q)$.

The lemma follows with $C = 4C_2$ since $\ell(c) = \ell(c') - d(p,p') - d(q,q')$ and $d(p,p') + d(q,q') = d(p,q) - d(p',q') = 4C_2$.

□

Let $b : X \to \mathbb{R}$ be a Busemann fuction based at ξ. For any subset $A \subset X$, let

$$H(A) := \sup\{b(x) | x \in A\} - \inf\{b(x) | x \in A\}$$

be the height change of points in A. Such a quantity can also be similarly defined for subsets of a Sol-type group since there is a notion of height in a Sol-type group.

Lemma 2.3. *Let X be a proper geodesic δ-hyperbolic space, $\xi \in \partial X$, b a Busemann function based at ξ, S a horosphere centered at ξ, and B the horoball with boundary S. Let $p, q \in S$ and $c : [0,l] \to X \backslash B$ a path with $c(0) = p$, $c(l) = q$. Then,*

1. *The length of c satisfies $\ell(c) \geq 2H(c) + 2^{\frac{d(p,q)-C-2}{2\delta}} - C - 5d(p,q)$.*
2. *Assume $H(c) > d(p,q)$. Then there are $0 \leq s < s' \leq t' < t \leq l$ such that $b(c(s)) = b(c(t)) < b(c(s')) = b(c(t'))$, $d(p,q) < |b(c(s)) - b(c(s'))| \leq 2d(p,q)$, and $\ell(c|_{[s,s']}) + \ell(c_{[t',t]}) \geq 2^{\frac{d(p,q)-C-2}{2\delta}} - C - d(p,q)$.*

Here C is the constant from Lemma 2.2, especially it only depends on δ.

Remark 2.4. Lemma 2.3 claims that c has a subpath (c itself if $H(c) \leq d(p,q)$ and $c|_{[s,s']}$ or $c_{[t',t]}$ if $H(c) > d(p,q)$) whose height change ($|b(c(s)) - b(c(s'))|$) is comparable with $d(p,q)$ but whose length is exponential in $d(p,q)$. This will be used in the proof of Lemma 4.7.

Proof. The lemma follows immediately from Lemma 2.2 when $H(c) \leq d(p,q)$. So we assume $H(c) > d(p,q)$. Let $b(p) = b_0 < b_1 < \cdots < b_m = b(p) + H(c)$ be such that $d(p,q) < b_{i+1} - b_i \leq 2d(p,q)$. Let $l_0 \in [0,l]$ be such that $c(l_0)$ is a lowest point on c, that is, $b(c(l_0)) = \max\{b(x) | x \in$

$c\}$. For each $1 \leq i < m$, let $s_i \in [0, l_0]$ be the last t in $[0, l_0]$ satisfying $b(c(t)) = b_i$ and similarly let $t_i \in [l_0, l]$ be the first t in $[l_0, l]$ satisfying $b(c(t)) = b_i$. We also set $s_0 = 0$, $t_0 = l$ and $s_m = t_m = l_0$. The choices of s_i and t_i imply that $c|_{[s_i, s_{i+1}]}$ and $c|_{[t_{i+1}, t_i]}$ lie below the horosphere $b = b_i$, that is, $b(c(t)) \geq b_i$ for $t \in [s_i, s_{i+1}] \cup [t_{i+1}, t_i]$. Let k be the integer such that $d(c(s_i), c(t_i)) \geq d(p, q)$ for all $i \leq k$ and $d(c(s_{k+1}), c(t_{k+1})) < d(p, q)$. Let γ be a geodesic between $c(s_{k+1})$ and $c(t_{k+1})$. Then the path $c|_{[s_k, s_{k+1}]} \cup \gamma \cup c|_{[t_{k+1}, t_k]}$ is a path below the horosphere $b = b_k$ joining $c(s_k)$ and $c(t_k)$. Now Lemma 2.2 implies

$$\ell(c|_{[s_k, s_{k+1}]}) + \ell(c_{[t_{k+1}, t_k]}) \geq 2^{\frac{d(p,q)-C-2}{2\delta}} - C - d(p, q).$$

Now for each $i \neq k$ by considering the height change we get $\ell(c|_{[s_i, s_{i+1}]}), \ell(c|_{[t_{i+1}, t_i]}) \geq |b_i - b_{i+1}|$. Now (1) follows as $\ell(c) = \sum_i (\ell(c|_{[s_i, s_{i+1}]}) + \ell(c|_{[t_{i+1}, t_i]}))$ and $\sum_i |b_i - b_{i+1}| = H(c)$. (2) holds with $s = s_k$, $s' = s_{k+1}$, $t' = t_{k+1}$, $t = t_k$.

□

2.3. **Heintze and Sol-type groups.** We defined Heintze groups and Sol-type groups briefly in the Introduction. They are built from a pair of data: a nilpotent Lie algebra and a derivation with special properties. In order to work in a given Heintze group or Sol-type group we need some tools which are a height function and a derivation that are not always defined canonically but that we would like to refer to without ambiguity; to do this we need to be more cautious, and fix some additional convention.

2.3.1. *Heintze groups.* Given a derivation D on a Lie algebra $\mathfrak{n}$, we denote by $\mathfrak{n} \rtimes_D \mathbb{R}$ the Lie algebra obtained as a semidirect product $\mathfrak{n} \rtimes \mathbb{R}$ where $1 \in \mathbb{R}$ acts on $\mathfrak{n}$ by the derivation D.

Definition 2.5 (Heintze group)**.** Let N be a nilpotent simply connected Lie group and let D be a derivation of $\mathfrak{n}$ that has only eigenvalues with positive real parts and the smallest one has real part equal to one. A *Heintze group* with normalized derivation D is a simply connected solvable Lie group having Lie algebra $\mathfrak{n} \rtimes_D \mathbb{R}$.

Heintze groups are Gromov-hyperbolic. Even better, they have at least one left-invariant Riemannian metric of negative sectional curvature [Hei74], and this is a characterization among connected Lie groups.

Definition 2.6 (Carnot-type Heintze group)**.** A Heintze group S with a normalized derivation D is of *Carnot type* if $\ker(D - 1)$ Lie generates $\mathfrak{n}$.

In the definition above, the property does not depend on the chosen normalized derivation D. Indeed, if one replaces D with $D' = D + \mathrm{ad}_{X_0}$ for some $X_0 \in \mathfrak{n}$, then D' induces the identity on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. It then follows from [Cor16, Lemma 3.10] that $\ker(D' - 1)$ again Lie generates $\mathfrak{n}$.

A Heintze group has a distinguished family of horospheres, disregarding the choice of a particular left-invariant Riemannian metric. Those are left cosets of the derived subgroup N. By focal point of a Heintze group we mean the limit point of the subgroup $N = [S, S]$ in the Gromov boundary. When S is naturally acting on its Gromov boundary, this point is the only one fixed by S.

Definition 2.7 (Real shadow)**.** Let D be a derivation of a real Lie algebra $\mathfrak{n}$. The derivation D may be decomposed into commuting components $D = D_{\mathsf{ss,r}} + D_{\mathsf{ss,i}} + D_{\mathsf{n}}$, where $D_{\mathsf{ss,r}}$ is semisimple with a real spectrum, $D_{\mathsf{ss,i}}$ is semisimple with purely imaginary spectrum, and D_{n} is nilpotent,

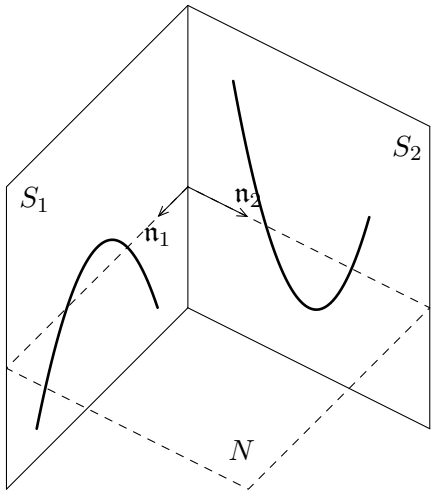

FIGURE 5. Sketch view of a Riemannian Sol-type group and two geodesics. Note that we do not assume that $\mathfrak{n}_1 \perp \mathfrak{n}_2$.

all being derivations ([LDG21, Corollary 2.6]). The real shadow of $\mathfrak{s} = \mathfrak{n} \rtimes_D \mathbb{R}$ is defined as $\mathfrak{s}_0 = \mathfrak{n} \rtimes_{(D_{\mathsf{ss},\mathsf{r}}+D_{\mathsf{n}})} \mathbb{R}$.

Heintze groups with a real shadow of Carnot type may be characterized geometrically by the fact that the conformal gauge on their boundary at infinity minus the focal point contains a geodesic metric, indeed even a subRiemannian one.

2.3.2. *Sol-type groups.* We define below a class of solvable groups, the most prominent of which is the three-dimensional group SOL.

Definition 2.8 (Sol-type)**.** Let N_1, N_2 be a pair of simply connected nilpotent Lie groups. Let $\lambda > 0$. Let D_1, D_2 be a pair of derivations of $\mathfrak{n}_1$ and $\mathfrak{n}_2$, respectively, so that $\mathfrak{n}_1 \rtimes_{D_1} \mathbb{R}$ and $\mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$ are the Lie algebras of two Heintze groups S_1 and S_2, i.e., the real parts of the eigenvalues of D_1, D_2 are positive and they are normalized so that the smallest ones of each have real parts equal to one. The derivation $D = D_1 \oplus (-\lambda D_2)$ acts on the Lie algebra $\mathfrak{n}_1 \times \mathfrak{n}_2$ and the corresponding semi-direct product

$$S = (N_1 \times N_2) \rtimes \mathbb{R}$$

is called a *Sol-type group*.

A Sol-type group is unimodular if and only if $\Re \operatorname{tr}(D_1) = \lambda \Re \operatorname{tr}(D_2)$ (which does not depend on D_1 and D_2 chosen).

Similar to SOL, the group G is foliated by the left cosets of $S_i = N_i \rtimes \mathbb{R}$. Note that S_2 is a "upside down" Heintze group, while S_1 is right side up. See Figure 5.

2.3.3. *Height.*

Definition 2.9. Let $S = N \rtimes_D \mathbb{R}$ be a Heintze group as in Definition 2.5. The projection $h\colon S \to \mathbb{R}$ is called the *height function* of S.

Definition 2.10. Let $G = N \rtimes_D \mathbb{R}$ be a Sol-type group as in Definition 2.8. The projection $h\colon G \to \mathbb{R}$ is called the *height function* of G.

In Definition 2.10 the height function does depend on a convention on the strucure of the group. For instance, in the group SOL, h and $-h$ are both height functions, depending on the choice of N_1 and N_2. When we consider a Sol-type group, the decomposition as $(N_1 \rtimes N_2) \rtimes_D \mathbb{R}$ will always be present in the background so that the height function is fixed.

3. Left-invariant Riemannian metrics on Heintze Groups

In this section we show that for every Heintze group $S = N \rtimes \mathbb{R}$, every two left-invariant Riemannian metrics on S are roughly similar through the identity map, see Theorem A.

Lemma 3.1. *Let S be a simply connected solvable Lie group and assume that $N := [S, S]$ has codimension 1 in S. For every left-invariant Riemannian metric g on S, there exists a one-parameter subgroup $c\colon S/N \to S$ that is a geodesic such that $\dot{c}(0) \perp \mathfrak{n}$ and $\pi \circ c$ is the identity on S/N, if $\pi\colon S \to S/N$ denotes the associated projection.*

Proof. Let ∇ be the Levi-Civita connection of the left-invariant metric g on S. Then by Koszul's formula for the Levi-Civita connection (see e.g. (5.3) in [Mil76]), for every $X, Y \in \mathfrak{s}$, $\nabla_X Y = \frac{1}{2}(\mathrm{ad}_X Y - \mathrm{ad}^*_X Y - \mathrm{ad}^*_Y X)$, where ad^*_X is such that $g(\mathrm{ad}^*_X Y, Z) = g(Y, \mathrm{ad}_X Z)$ for all Y, Z. It follows that the one-parameter subgroup c generated by T with $T \in \mathfrak{n}^\perp$ and $g(T, T) = 1$ is a geodesic of g, since $\nabla_{\dot{c}}\dot{c} = 0$. □

The key of the proof of Theorem A is Lemma 2.1 from the previous section and the fact that for every two one-parameter subgroups c_1, c_2 of S not contained in N, every left coset of $c_1(\mathbb{R})$ is at bounded distance from a unique left coset of $c_2(\mathbb{R})$, see Lemma 3.2.

Let g_1, g_2 be two left-invariant Riemannian metrics on a Heintze group S. Let c_1 and c_2 be the one-parameter subgroups associated to g_1 and g_2 respectively by Lemma 3.1.

In the case when c_1 and c_2 have the same image, the rest of the proof of Theorem A is quite simple. We shall treat this case in the next section; afterwards we consider the general case.

Proof of Theorem A when c_1 and c_2 have the same image. Observe that the height map $S \to \mathbb{R}$ is 1-Lipschitz, where we equip S with d_i and S/N with the Hausdorff distance $\mathrm{Hausdist}_{d_i}$ for $i = 1, 2$. From now on we decompose S topologically as a product $N \times \mathbb{R}$ where $c_1(t) = (1_N, t)$ for all $t \in \mathbb{R}$, and for all $n \in N$ we denote $\mathfrak{c}_n$ the curve $\mathfrak{c}_n(t) = (n, t)$. By rescaling the metric g_1 and g_2 we may assume that $c_1 = c_2$, and that they are unit-speed geodesics for d_1 and d_2. It follows from the normalization convention that for $i = 1, 2$, $\mathrm{Hausdist}_{d_i}$ on S/N is also the standard absolute value on $\mathbb{R}$. A useful consequence is that if two subsets are at d_i-Hausdorff distance bounded by H for some i, then so are their maximal heights also differ by H.

Let C be the constant from Lemma 2.1 for both d_1 and d_2. We shall show that the identity map $\mathrm{Id} : (S, d_1) \to (S, d_2)$ is a rough isometry. Let $x = (n, t), \tilde{x} = (\tilde{n}, \tilde{t}) \in S$. Our assumption implies that the curves $\mathfrak{c}_n$ and $\mathfrak{c}_{\tilde{n}}$ are unit-speed minimizing geodesics with respect to both d_1 and d_2. Because of Lemma 2.1, for each $i = 1, 2$ exists t_i such that the path $\beta_i := x\mathfrak{c}_n(t_i) \cup \mathfrak{c}_n(t_i)\mathfrak{c}_{\tilde{n}}(t_i) \cup \mathfrak{c}_{\tilde{n}}(t_i)y$ is a $(1, C)$-quasi-geodesic in (S, d_i) from x to y. Since the identity map $(S, d_1) \to (S, d_2)$ is biLipschitz, the path β_2 is an (L, A)-quasi-geodesic in (S, d_1) from x to y, where L, A depend only on d_1 and d_2. By the Morse Lemma, the Hausdorff distance between β_1 and β_2 in (S, d_1) is bounded above by a constant depending only on d_1 and d_2. Comparing

heights we see that $|t_1 - t_2|$ is bounded above by a constant depending only on d_1 and d_2. Finally Lemma 2.1 implies that $|d_1(x,y) - d_2(x,y)|$ is bounded above by a constant depending only on d_1 and d_2. This finishes the proof of Theorem A when c_1 and c_2 have the same image. □

3.1. **The general case: c_1 and c_2 might have different images.** In order to consider the general case in the proof of Theorem A, we need the following lemma. We shall abbreviate the image of $\mathbb{R}$ under a one-parameter subgroup $c\colon \mathbb{R} \to S$ by c.

Lemma 3.2. *Let S be a Heintze group with derived subgroup N. Equip S with a left-invariant Riemannian metric g. For every two one-parameter subgroups c_1, c_2 of S not contained in N, there is a positive number C (depending on c_1, c_2 and g) such that for every $s_1 \in S$, there is a unique left coset s_2c_2 of c_2, with $s_2 \in S$, such that*

$$\mathrm{Hausdist}_d(s_1c_1, s_2c_2) \leq C,$$

where $\mathrm{Hausdist}_d$ *denotes the Hausdorff distance with respect to the distance d on S determined by g.*

Proof. Let g_2 be a left-invariant Riemannian metric on S such that c_2 is orthogonal to N with respect to g_2, and denote d_2 the associated Riemannian distance. For every $s_1 \in S$, the curve s_1c_1 is an (L,C)-quasi-geodesic in (S,d_2) for some constants L, C depending only on g and g_2. By the Morse Lemma, there is a complete geodesic γ in (S,d_2) such that $\mathrm{Hausdist}_{d_2}(s_1c_1,\gamma) \leq C_1$ for some constant C_1 depending only on g and g_2. Since s_1c_1 intersects all the horospheres centered at the focal point, so does γ (Indeed, $h(s_1c_1)$ and $h(\gamma)$ are both intervals of $\mathbb{R}$ at bounded Hausdorff distance from each other, so if one of them is $\mathbb{R}$ so is the other one). We see that the limit points of γ in ∂S are the focal point and some $n \in N$. On the other hand, there is a left coset s_2c_2 with the same limit points. Since both γ and s_2c_2 are geodesics in (S,d_2), their Hausdorff distance is bounded above by a constant H depending only on d_2. Hence $\mathrm{Hausdist}_{d_2}(s_1c_1, s_2c_2) \leq C_1 + H$ for some left coset s_2c_2 of c_2. The lemma follows as all the left-invariant Riemannian metrics are biLipschitz with respect to each other. Since two different cosets s_2c_2 and $s_2'c_2$ have infinite Hausdorff distance, we have uniqueness. □

Proof of Theorem A in the general case. For each $i \in \{1,2\}$, let g_i be a left-invariant Riemannian metric on S and d_i the distance on S determined by g_i. We need to show that the identity map $(S,d_1) \to (S,d_2)$ is a rough similarity. Let c_i be a g_i-geodesic section of $\pi\colon S \to S/N$ with $c_i(+\infty)$ equal to the focal point for all i. The composition $h \circ c_i : \mathbb{R} \to \mathbb{R}$ is the identity map. After rescaling the metric g_i if necessary, we may further assume that c_i is a unit-speed geodesic in (S,d_i). We shall show that the identity map $(S,d_1) \to (S,d_2)$ is a rough isometry. By symmetry it suffices to show that there is a constant C such that $d_1(x,y) \leq d_2(x,y) + C$ for every $x, y \in S$.

By Lemma 2.1, there are points $x' \in xc_2$, $y' \in yc_2$ such that $d_2(x',y') \leq C'$ and

$$|d_2(x,y) - (d_2(x,x') + d_2(y',y))| \leq C', \tag{3.1}$$

where C' depends only on d_2. Since $\mathrm{Id} : (S,d_1) \to (S,d_2)$ is L-biLipschitz for some $L \geq 1$, we have

$$d_1(x',y') \leq LC'. \tag{3.2}$$

By Lemma 3.2 there are left cosets α, β of c_1 such that $\mathrm{Hausdist}_{d_1}(\alpha, xc_2) \leq C$, $\mathrm{Hausdist}_{d_1}(\beta, yc_2) \leq C$, where C is a constant depending only on d_1, d_2.

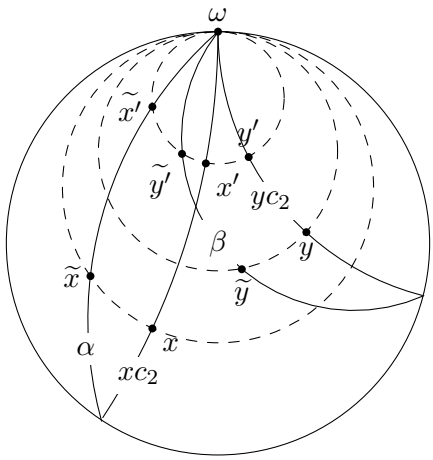

FIGURE 6. Main objects in the proof of Theorem A in the general case, in the hyperbolic disk model with a focal point ω. From the point of view of d_2, d_1-geodesics appear in the form of hypercircles.

Considering the height function h, we take $\tilde{x}$ and $\tilde{x}'$ to be points on α satisfying $h(\tilde{x}) = h(x)$, $h(\tilde{x}') = h(x')$, see Figure 6. Similarly let $\tilde{y}$ and $\tilde{y}'$ be points on β satisfying $h(\tilde{y}) = h(y)$, $h(\tilde{y}') = h(y')$. We claim that we have

$$d_1(z, \tilde{z}) \leq 2C, \qquad \text{for } z \in \{x, x', y, y'\} \text{ and the respective } \tilde{z}. \tag{3.3}$$

Indeed, the d_1 distance from z to the appropriate left coset of c_1 is at most C, so that the height of the nearest-point projection of z on this left coset differs at most C from that of $\tilde{z}$.

We have the bounds

$$\begin{aligned} d_1(x,y) &\leq d_1(x,\tilde{x}) + d_1(\tilde{x},\tilde{x}') + d_1(\tilde{x}',x') + d_1(x',y') \\ &\quad + d_1(y',\tilde{y}') + d_1(\tilde{y}',\tilde{y}) + d_1(\tilde{y},y) \\ &\leq 8C + LC' + d_1(\tilde{x},\tilde{x}') + d_1(\tilde{y}',\tilde{y}) \\ &= 8C + LC' + d_2(x,x') + d_2(y',y) \\ &\leq 8C + LC' + C' + d_2(x,y), \end{aligned}$$

where we used the following arguments: In the first line, we used the triangle inequality. In the second line, we used (3.2) and (3.3). In the third line, we used that $d_1(\tilde{x},\tilde{x}') = |h(\tilde{x}) - h(\tilde{x}')| = |h(x) - h(x')| = d_2(x,x')$ and similarly, $d_1(\tilde{y},\tilde{y}') = d_2(y,y')$. In the fourth line, we used (3.1). □

4. Left-invariant Riemannian metrics on Sol-type groups

In this section we show that every two left-invariant Riemannian metrics with associated distances d_1 and d_2 on a Sol-type group G are roughly similar through the identity map, see Theorem B. Notation here is as in § 2.3.2, especially $G := (N_1 \times N_2) \rtimes \mathbb{R}$, $S_1 := N_1 \rtimes \mathbb{R}$, and $S_2 := N_2 \rtimes \mathbb{R}$.

The general strategy is the same as for the Heintze groups: we first establish the statement for those pairs of Riemannian metrics for which $\mathfrak{n}_1 \times \mathfrak{n}_2$ have the same orthogonal complement

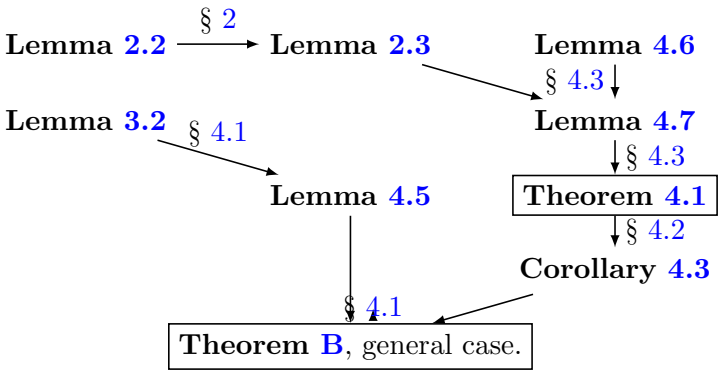

FIGURE 7. Scheme of the proof of Theorem B

in $\mathfrak{g}$, then the general case. To establish the special case we need to find an estimate for the distance function, see Theorem 4.1.

4.1. **Distances on Sol-type groups and the proof of Theorem B.** In this subsection we state a result giving an estimate of distances in Sol-type groups and use it to prove Theorem B. The estimate itself will be established later.

Let g be a left-invariant Riemannian metric on G. By Lemma 3.1 we choose a geodesic section for $G \to \mathbb{R}$, and then assume without loss of generality that as a set, $G = N_1 \times N_2 \times \mathbb{R}$, where the $\mathbb{R}$ direction is perpendicular to both N_1 and N_2 with respect to g. As in the case of Heintze groups, this assumption implies that for every $x \in N_1$, $y \in N_2$, the curve $\gamma_{x,y}(t) = (x, y, t)$ $(t \in \mathbb{R})$ is a minimizing constant-speed geodesic. These will be called *vertical geodesics*.

We define several maps. The map $h : G \to \mathbb{R}$, $h(x, y, t) := t$, will be called the height function of G and t will be called the height of the point (x, y, t). The "projections" $\pi_i : G \to S_i$ are defined by $\pi_1(x, y, t) := (x, t)$, $\pi_2(x, y, t) := (y, t)$. We emphasize that the maps π_i are not nearest point projections. However, they are Lie group homomorphisms and so are Lipschitz with respect to left-invariant Riemannian metrics, see Lemma 4.6.

By rescaling the metric g we may assume that the vertical geodesics are unit-speed geodesics. Denote by d the distance on G determined by g. We identify S_1 with $N_1 \times \{0\} \times \mathbb{R} \subset G$ and S_2 with $\{0\} \times N_2 \times \mathbb{R} \subset G$. For $j = 1, 2$, let $g^{(j)}$ be the Riemannian metric on S_j induced by g and $d^{(j)}$ the associated distance on S_j.

Define a "distance" $\rho : G \times G \to [0, \infty)$ on G by:

$$\rho(p, q) = d^{(1)}(\pi_1(p), \pi_1(q)) + d^{(2)}(\pi_2(p), \pi_2(q)) - |h(p) - h(q)|. \tag{4.1}$$

It turns out that the distance d on G differs from ρ by a bounded constant:

Theorem 4.1. *Let G, d and ρ be as above. Then there exists a constant $C > 0$ such that $|d(p, q) - \rho(p, q)| \leq C$ for all $p, q \in G$.*

Remark 4.2. Some care is needed in using Theorem 4.1. Given an arbitrary left-invariant Riemannian metric g on $S = (N_1 \times N_2) \rtimes \mathbf{R}$ we first pick a one-parameter subgroup A of S that is perpendicular to $N_1 \times N_2$ with respect to g and write S as $S = (N_1 \times N_2) \rtimes A$. The left cosets of A are the vertical geodesics. In other words, the maps π_i depend on the metric g.

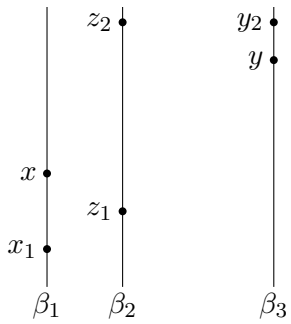

FIGURE 8. The left cosets of c in Corollary 4.3.

Theorem 4.1 will be proved in Section 4.3. As a consequence of the proof of Theorem 4.1 we will have

Corollary 4.3. Let G be a Sol-type group, g a left-invariant Riemannian metric, and d the associated distance. Then there is a constant $C > 0$ with the following property. Denote by c the one-parameter subgroup of G that is orthogonal to $N_1 \times N_2$ at e with respect to g. For every $x, y \in G$ with $h(x) \leq h(y)$, there exist three left cosets β_j $(j = 1, 2, 3)$ of c with $x \in \beta_1$, $y \in \beta_3$ and points $x_1 \in \beta_1$, $z_1, z_2 \in \beta_2$ and $y_2 \in \beta_3$ satisfying:

(1) $|d(x, y) - (d(x, x_1) + d(z_1, z_2) + d(y_2, y))| \leq C$;
(2) $d(x_1, z_1) \leq C$, $d(z_2, y_2) \leq C$;
(3) $h(x_1) = h(z_1) \leq h(x)$, $h(z_2) = h(y_2) \geq h(y)$.

Remark 4.4. Tom Ferragut has a result similar to Theorem 4.1, see Corollary 4.17 of [Fer20]. These two results have overlap but do not imply each other. The result in [Fer20] is for horospherical products $X \bowtie Y$ of Gromov hyperbolic Busemann spaces X, Y. On the one hand, horospherical products are more general than Sol-type groups. On the other hand, the factors X and Y in a horospherical product are "perpendicular" in some sense, while N_1 and N_2 in a Sol-type group is not assumed to be perpendicular to each other with respect to the metric g (without loss of generality the direction of the $\mathbb{R}$ factor is perpendicular to both N_1 and N_2, but we do not assume that N_1 and N_2 are perpendicular).

Proof of Theorem B assuming Theorem 4.1 in the case of equal vertical geodesics. One may decompose G as a product, $G = N_1 \times N_2 \times \mathbb{R}$, in such a way that the direction of the $\mathbb{R}$ factor is perpendicular to both N_1 and N_2 with respect to g_1 and g_2.

After rescaling we may further assume that vertical geodesics have unit speed with respect to both g_1 and g_2. We shall show that the identity map $\mathrm{Id} : (G, d_1) \to (G, d_2)$ is a rough isometry.

Let $g_i^{(j)}$ be the Riemannian metric on S_j induced by g_i, and let $d_i^{(j)}$ be the associated distance on S_j. Also let ρ_i be the "distance" (see (4.1)) on G corresponding to g_i. By Theorem 4.1 there is a constant $C > 0$ such that $|d_i(p, q) - \rho_i(p, q)| \leq C$ for all $p, q \in G$. On the other hand, since the vertical geodesics have unit speed in (S, g_i), Theorem A implies $|d_1^{(j)}(\pi_j(p), \pi_j(q)) - d_2^{(j)}(\pi_j(p), \pi_j(q))| \leq C'$ for some constant $C' \geq 0$ and all $p, q \in G$. It follows from the definition of ρ_i that $|\rho_1(p, q) - \rho_2(p, q)| \leq 2C'$ and so $|d_1(p, q) - d_2(p, q)| \leq 2C + 2C'$ for all $p, q \in G$. □

For the general case, we need an analogue of Lemma 3.2 for Sol-type groups.

Lemma 4.5. *Let $c, \tilde{c} : \mathbb{R} \to G = (N_1 \times N_2) \rtimes \mathbb{R}$ be one-parameter subgroups of G not contained in $(N_1 \times N_2) \times \{0\} \subset G$. Let g be any left-invariant Riemannian metric on G. Then there is a constant C such that for any left coset pc of c, there is a unique left coset $q\tilde{c}$ of $\tilde{c}$ such that* $\mathrm{Hausdist}_d(pc, q\tilde{c}) \leq C$.

Proof. We first establish the existence. Since $c, \tilde{c}$ are not contained in $(N_1 \times N_2) \times \{0\}$, the compositions $h \circ c$ and $h \circ \tilde{c}$ are automorphisms of $\mathbb{R}$. By composing c and $\tilde{c}$ with suitable automorphisms of $\mathbb{R}$ we may assume $h \circ c(t) = t$ and $h \circ \tilde{c}(t) = t$ for $t \in \mathbb{R}$. The one-parameter subgroups c and $\tilde{c}$ now have the expressions: $c(t) = (a_1(t), a_2(t), t)$ and $\tilde{c}(t) = (\tilde{a}_1(t), \tilde{a}_2(t), t)$ for some functions $a_1, \tilde{a}_1 : \mathbb{R} \to N_1$, $a_2, \tilde{a}_2 : \mathbb{R} \to N_2$.

As any two left-invariant Riemannian distances on S are biLipschitz equivalent, we may assume $\tilde{c}$ is perpendicular to $N_1 \times N_2$ and is unit speed with respect to g. Then $\tilde{c}(t_1)$ and $\tilde{c}(t_2)$ realize the distance between $(N_1 \times N_2) \times \{t_1\}$ and $(N_1 \times N_2) \times \{t_2\}$. Since the distance d is left invariant, we may assume $p = e$ and so $pc = c$. As the projections π_1 and π_2 are group homomorphisms, the compositions $\pi_i \circ c$, $\pi_i \circ \tilde{c}$ are one-parameter subgroups of S_i that are not contained in $N_i \times \{0\} \subset S_i$. By Lemma 3.2, there exist constants $C_1, C_2 > 0$ depending only on c, $\tilde{c}$ and g, $(n_1, 0) \in S_1$ and $(n_2, 0) \in S_2$ such that $\mathrm{Hausdist}_{d_1}(\pi_1(c), (n_1, 0)\pi_1(\tilde{c})) \leq C_1$ and $\mathrm{Hausdist}_{d_2}(\pi_2(c), (n_2, 0)\pi_2(\tilde{c})) \leq C_2$, where d_i denotes the distance on S_i induced by the restriction of g to S_i. Here we identify S_1 with $N_1 \times \{0\} \times \mathbb{R} \subset G$ and similarly S_2 with $\{0\} \times N_2 \times \mathbb{R} \subset G$. Clearly we have $d(x, y) \leq d_i(x, y)$ for all $x, y \in S_i$. We claim that $d_1((a_1(t), t), (n_1\tilde{a}_1(t), t)) \leq 2C_1$ for all $t \in \mathbb{R}$. To see this, let $t \in \mathbb{R}$. Since $\mathrm{Hausdist}_{d_1}(\pi_1(c), (n_1, 0)\pi_1(\tilde{c})) \leq C_1$, there is some $(n_1\tilde{a}_1(t'), t') \in (n_1, 0)\pi_1(\tilde{c})$ such that $d_1((a_1(t), t), (n_1\tilde{a}_1(t'), t')) \leq C_1$. As the distance between $(N_1 \times N_2) \times \{t_1\}$ and $(N_1 \times N_2) \times \{t_2\}$ is $|t_1 - t_2|$, we have $|t - t'| \leq C_1$. Now the triangle inequality implies $d_1((a_1(t), t), (n_1\tilde{a}_1(t), t)) \leq 2C_1$. Similarly $d_2((a_2(t), t), (n_2\tilde{a}_2(t), t)) \leq 2C_2$ for all $t \in \mathbb{R}$.

Set $q = (n_1, n_2, 0) \in G$. We next show that $d(c(t), q\tilde{c}(t)) \leq 2C_1 + 2C_2$ for all $t \in \mathbb{R}$ and so $\mathrm{Hausdist}_d(c, q\tilde{c}) \leq 2C_1 + 2C_2$.

$$
\begin{aligned}
&d(c(t), q\tilde{c}(t)) \\
&= d((a_1(t), a_2(t), t), (n_1, n_2, 0)(\tilde{a}_1(t), \tilde{a}_2(t), t)) \\
&\leq d((a_1(t), a_2(t), t), (n_1\tilde{a}_1(t), a_2(t), t)) + d((n_1\tilde{a}_1(t), a_2(t), t), (n_1, n_2, 0)(\tilde{a}_1(t), \tilde{a}_2(t), t)) \\
&= d((0, a_2(t), 0)(a_1(t), 0, t), (0, a_2(t), 0)(n_1\tilde{a}_1(t), 0, t)) + \\
&+ d((n_1\tilde{a}_1(t), 0, 0)(0, a_2(t), t), (n_1\tilde{a}_1(t), 0, 0)(0, n_2\tilde{a}_2(t), t)) \\
&= d((a_1(t), 0, t), (n_1\tilde{a}_1(t), 0, t)) + d((0, a_2(t), t), (0, n_2\tilde{a}_2(t), t)) \\
&\leq d_1(a_1(t), t), (n_1\tilde{a}_1(t), t)) + d_2((a_2(t), t), (n_2\tilde{a}_2(t), t)) \\
&\leq 2C_1 + 2C_2.
\end{aligned}
$$

The uniqueness follows by applying the uniqueness claim in Lemma 3.2 to $\pi_i(c)$ and $\pi_i(\tilde{c})$. □

Proof of Theorem B in the general case. Let $g, \tilde{g}$ be left-invariant Riemannian metrics on G and $d, \tilde{d}$ the distances on G determined by g, $\tilde{g}$ respectively. We need to show that the identity map $(G, d) \to (G, \tilde{d})$ is a rough similarity.

Let c be the one-parameter subgroup of G whose tangent vector at e is g-perpendicular to $(N_1 \times N_2) \times \{0\}$ and $h(c(t)) = t$. After rescaling the metric g if necessary we may assume that c

is a unit-speed geodesic with respect to g. This normalization implies that if β is a left coset of c and $p_1, p_2 \in \beta$, then $d(p_1, p_2) = |h(p_1) - h(p_2)|$. Similarly let $\tilde{c}$ be the normalized one-parameter subgroup of G corresponding to $\tilde{g}$ such that $h(\tilde{c}(t)) = t$. We observe that, if β is a left coset of c and $\tilde{\beta}$ is a left coset of $\tilde{c}$, and $p_1, p_2 \in \beta$, $\tilde{p}_1, \tilde{p}_2 \in \tilde{\alpha}$ with $h(\tilde{p}_1) = h(p_1)$, $h(\tilde{p}_2) = h(p_2)$, then $d(p_1, p_2) = |h(p_1) - h(p_2)| = \tilde{d}(\tilde{p}_1, \tilde{p}_2)$. We shall show that the identity map $(G, d) \to (G, \tilde{d})$ is a rough isometry.

By symmetry it suffices to show that there is a constant C such that $d(p, q) \le \tilde{d}(p, q) + C$ for every $p, q \in G$. Let $p, q \in G$. We may assume $h(p) \le h(q)$. By Corollary 4.3, there are three left cosets $\tilde{\beta}_i$ $(i = 1, 2, 3)$ of $\tilde{c}$ with $p \in \tilde{\beta}_1$, $q \in \tilde{\beta}_3$, points $\tilde{p}_1 \in \tilde{\beta}_1$, $\tilde{r}_1, \tilde{r}_2 \in \tilde{\beta}_2$, $\tilde{q}_2 \in \tilde{\beta}_3$ satisfying the following (setting $\tilde{p}_2 := p$, $\tilde{q}_1 := q$): $h(\tilde{p}_1) = h(\tilde{r}_1) \le h(\tilde{p}_2)$, $h(\tilde{r}_2) = h(\tilde{q}_2) \ge h(\tilde{q}_1)$, $\tilde{d}(\tilde{p}_1, \tilde{r}_1) < \tilde{C}$, $\tilde{d}(\tilde{r}_2, \tilde{q}_2) < \tilde{C}$, and $|\tilde{d}(p, q) - (\tilde{d}(p, \tilde{p}_1) + \tilde{d}(\tilde{r}_1, \tilde{r}_2) + \tilde{d}(\tilde{q}_2, q))| \le \tilde{C}$, where $\tilde{C}$ is a constant depending only on $\tilde{d}$. By Lemma 4.5, there are left cosets β_i $(i = 1, 2, 3)$ of c, such that for every $x \in \beta_i$, $y \in \tilde{\beta}_i$ with the same height (that is, $h(x) = h(y)$) we have $d(x, y) \le C$, where C is a constant depending only on d and $\tilde{c}$. Let $p_j \in \beta_1$, $r_j \in \beta_2$, $q_j \in \beta_3$ $(j = 1, 2)$ satisfying $h(p_j) = h(\tilde{p}_j)$, $h(r_j) = h(\tilde{r}_j)$, $h(q_j) = h(\tilde{q}_j)$. Since d and $\tilde{d}$ are biLipschitz through the identity, there is a constant C_1 depending only on d, $\tilde{d}$ such that $d(\tilde{p}_1, \tilde{r}_1), d(\tilde{r}_2, \tilde{q}_2) \le C_1$. Now we have

$$\begin{aligned}
d(p, q) &= d(\tilde{p}_2, \tilde{q}_1) \\
&\le d(\tilde{p}_2, p_2) + d(p_2, p_1) + d(p_1, \tilde{p}_1) + d(\tilde{p}_1, \tilde{r}_1) + d(\tilde{r}_1, r_1) \\
&\quad + d(r_1, r_2) + d(r_2, \tilde{r}_2) + d(\tilde{r}_2, \tilde{q}_2) + d(\tilde{q}_2, q_2) + d(q_2, q_1) + d(q_1, \tilde{q}_1) \\
&\le d(p_2, p_1) + d(r_1, r_2) + d(q_2, q_1) + 6C + 2C_1 \\
&= \tilde{d}(\tilde{p}_2, \tilde{p}_1) + \tilde{d}(\tilde{r}_1, \tilde{r}_2) + \tilde{d}(\tilde{q}_2, \tilde{q}_1) + 6C + 2C_1 \\
&\le \tilde{d}(\tilde{p}_2, \tilde{q}_1) + \tilde{C} + 6C + 2C_1 \\
&= \tilde{d}(p, q) + \tilde{C} + 6C + 2C_1.
\end{aligned}$$

□

4.2. **Another expression for ρ.** We next start the proof of Theorem 4.1 and Corollary 4.3. Up to an additive constant, the function ρ as in (4.1) admits another expression which is more convenient for our purpose. We first fix some notation.

Let G, g, d, $d^{(j)}$ and ρ be as in Subsection 4.1. We recall that g is a left-invariant Riemannian metric on G such that the $\mathbb{R}$ direction is perpendicular to both N_1 and N_2 with respect to g, the vertical geodesics γ_{x_1,x_2} $(x_1 \in N_1, x_2 \in N_2)$ are unit-speed minimizing geodesics, and the minimal distance between the two "horizontal sets" $N_1 \times N_2 \times \{t_1\}$ and $N_1 \times N_2 \times \{t_2\}$ is $|t_1 - t_2|$.

Since S_1, S_2 are Gromov-hyperbolic, there is some constant $\delta > 0$ such that both $(S_1, d^{(1)})$ and $(S_2, d^{(2)})$ are δ-hyperbolic.

For each $t \in \mathbb{R}$, let $d_t^{(1)}$ be the path metric on the set $N_1 \times \{t\} \subset (S_1, d^{(1)})$. By the geometry of Heintze groups, we know that for fixed $x_1, y_1 \in N_1$, the quantity $d_t^{(1)}(\gamma_{x_1}(t), \gamma_{y_1}(t))$ decreases exponentially as $t \to +\infty$. Let $t_{x_1,y_1} \in \mathbb{R}$ be such that $d^{(1)}_{t_{x_1,y_1}}(\gamma_{x_1}(t_{x_1,y_1}), \gamma_{y_1}(t_{x_1,y_1})) = 1$. Define a function $\tilde{\rho}_1 : S_1 \times S_1 \to [0, +\infty)$ by:

$$\tilde{\rho}_1((x_1, t), (y_1, s)) = \begin{cases} |t - s| + 1 & \text{if } t_{x_1,y_1} \le \max\{t, s\} \\ (t_{x_1,y_1} - t) + (t_{x_1,y_1} - s) + 1 & \text{if } t_{x_1,y_1} > \max\{t, s\}. \end{cases}$$

The quantity $\tilde{\rho}_1((x_1, t), (y_1, s))$ is (roughly) the length of a path in S_1 between (x_1, t) and (y_1, s). To simplify notation, denote $p = (x_1, t)$, $q = (y_1, s)$. First assume $t_{x_1,y_1} > \max\{t, s\}$.

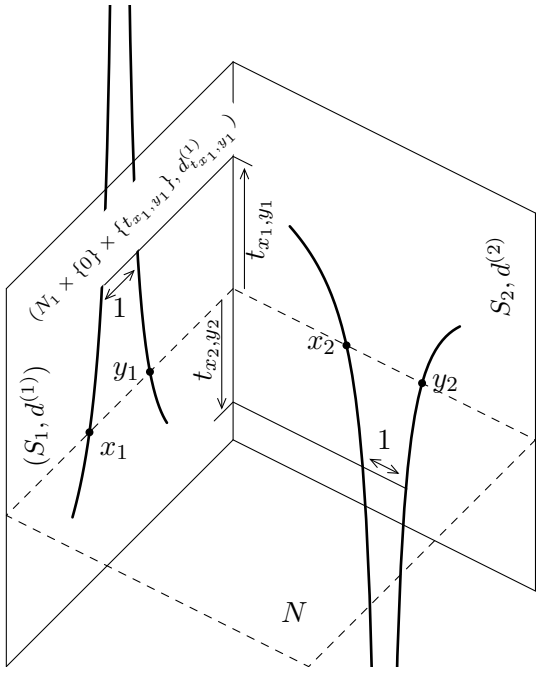

FIGURE 9. "Metric view" of the vertical geodesics and definition of t_{x_1,y_1} in a Sol-type group. Beware that this is not a coordinate view and we equip left cosets of subgroups with path (Riemannian) metrics.

Let $\tilde{\gamma}_{pq} := \gamma_{x_1}|_{[t,t_{x_1,y_1}]} * c * \bar{\gamma}_{y_1}|_{[s,t_{x_1,y_1}]}$ be the concatenation of three paths, where $c : [0,1] \to N_1 \times \{t_{x_1,y_1}\}$ is a path in the horosphere $N_1 \times \{t_{x_1,y_1}\}$ with length 1 from (x_1, t_{x_1,y_1}) to (y_1, t_{x_1,y_1}). Here for every curve $\alpha : [a,b] \to X$ in a space X, we will use $\bar{\alpha} : [a,b] \to X$ to denote the curve $\bar{\alpha}(t) = \alpha(a+b-t)$ with the same image as that of α but reverse orientation. It is clear that $\ell(\tilde{\gamma}_{pq}) = \tilde{\rho}_1(p,q)$. Next assume $t_{x_1,y_1} \leq \max\{t,s\}$. Without loss of generality we may assume $t < s$. In this case, let $\tilde{\gamma}_{pq} := \gamma_{x_1}|_{[t,s]} * c$, where c is a minimal length path in the horosphere $N_1 \times \{s\}$ from (x_1, s) to (y_1, s). It is clear that $\tilde{\rho}_1(p,q) - 1 \leq \ell(\tilde{\gamma}_{pq}) \leq \tilde{\rho}_1(p,q)$.

It is easy to see that the path $\tilde{\gamma}_{pq}$ is a $(1,H)$ quasigeodesic between p and q and that its length $\ell(\tilde{\gamma}_{pq}) \leq d^{(1)}(p,q) + H$, with $H \geqslant 0$ depending only on δ. Hence by stability of quasigeodesics in Gromov-hyperbolic spaces, for every length-minimizing geodesic γ_{pq} between p and q the Hausdorff distance between $\tilde{\gamma}_{pq}$ and γ_{pq} satisfies:

$$\text{Hausdist}^{(1)}(\tilde{\gamma}_{pq}, \gamma_{pq}) \leq C, \tag{4.2}$$

where C depends only on δ.

Similarly, let $d^{(2)}_t$ be the path metric on the set $N_2 \times \{t\} \subset (S_2, d^{(2)})$. For fixed $x_2, y_2 \in N_2$, the quantity $d^{(2)}_t(\gamma_{x_2}(t), \gamma_{y_2}(t))$ decreases exponentially as $t \to -\infty$. Let $t_{x_2,y_2} \in \mathbb{R}$ be such that $d^{(2)}_{t_{x_2,y_2}}(\gamma_{x_2}(t_{x_2,y_2}), \gamma_{y_2}(t_{x_2,y_2})) = 1$. Define a function $\tilde{\rho}_2 : S_2 \times S_2 \to [0, +\infty)$ by:

$$\tilde{\rho}_2((x_2,t),(y_2,s)) = \begin{cases} |t-s| + 1 & \text{if } t_{x_2,y_2} \geq \min\{t,s\} \\ (t - t_{x_2,y_2}) + (s - t_{x_2,y_2}) + 1 & \text{if } t_{x_2,y_2} < \min\{t,s\}. \end{cases}$$

There is a constant $C > 0$ such that

$$|d^{(j)}((x_j,t),(y_j,s)) - \tilde{\rho}_j((x_j,t),(y_j,s))| \leq C, \ \ \forall (x_j,t),(y_j,s) \in S_j.$$

Define $\tilde{\rho} : G \times G \to [0, +\infty)$ by

$$\tilde{\rho}(p, q) = \tilde{\rho}_1(\pi_1(p), \pi_1(q)) + \tilde{\rho}_2(\pi_2(p), \pi_2(q)) - |h(p) - h(q)|.$$

Then ρ and $\tilde{\rho}$ differ by at most a fixed constant.

Let $p, q \in G$ and write $p = (x_1, x_2, t)$ and $q = (y_1, y_2, s)$. Without loss of generality we may assume that $s \geq t$. We shall construct a path α_0 from q to p. We first notice that the left cosets of S_i equipped with the path metric is isometric to $(S_i, d^{(i)})$. Denote $p' = (x_1, y_2, t)$. Let $\alpha_0 = \tilde{\gamma}_{qp'} * \tilde{\gamma}_{p'p}$, where $\tilde{\gamma}_{qp'} \subset N_1 \times \{y_2\} \times \mathbb{R} \approx S_1$ is the path from q to p' constructed in this subsection and similarly $\tilde{\gamma}_{p'p} \subset \{x_1\} \times N_2 \times \mathbb{R} \approx S_2$ is the path from p' to p. We have $\tilde{\rho}_1((y_1, s), (x_1, t)) - 1 \leq \ell(\tilde{\gamma}_{qp'}) \leq \tilde{\rho}_1((y_1, s), (x_1, t))$. Let $q' = (x_1, y_2, s)$. Then $\tilde{\gamma}_{q'p}$ is the concatenation of $\bar{\gamma}_{x_1}|_{[t,s]}$ and $\tilde{\gamma}_{p'p}$. Hence we have $\tilde{\rho}_2(((x_2, t), (y_2, s)) - 1 \leq \ell(\tilde{\gamma}_{p'p}) + |s - t| \leq \tilde{\rho}_2(((x_2, t), (y_2, s))$. It follows that the length ℓ_0 of α_0 satisfies $|\ell_0 - \tilde{\rho}(p, q)| \leq 2$.

Since ρ, $\tilde{\rho}$, ℓ_0 differ from each other by a fixed constant, the following gives an expression for these quantities up to a constant: for $p = (x_1, x_2, t)$, $q = (y_1, y_2, s)$:

$$\tilde{\rho}(p, q) = \begin{cases} |t - s| + 2 & \text{if } t_{x_2,y_2} \geq \min\{t, s\} \text{ and } t_{x_1,y_1} \leq \max\{t, s\} \\ 2t_{x_1,y_1} - (s + t) + 2 & \text{if } t_{x_2,y_2} \geq \min\{t, s\} \text{ and } t_{x_1,y_1} \geq \max\{t, s\} \\ (s + t) - 2t_{x_2,y_2} + 2 & \text{if } t_{x_2,y_2} \leq \min\{t, s\} \text{ and } t_{x_1,y_1} \leq \max\{t, s\} \\ 2t_{x_1,y_1} - 2t_{x_2,y_2} - |s - t| + 2 & \text{if } t_{x_2,y_2} \leq \min\{t, s\} \text{ and } t_{x_1,y_1} \geq \max\{t, s\}. \end{cases}$$

Proof of Corollary 4.3 assuming Theorem 4.1. We first consider the case when $\mathbb{R}$ is perpendicular to N with respect to g. In this case $c = \{0\} \times \{0\} \times \mathbb{R}$ and the left cosets of c are vertical geodesics $\gamma_{x,y}$. Let $p = (x_1, x_2, t)$ and $q = (y_1, y_2, s)$ with $t \leq s$. With the notation from above, the three left cosets are $\beta_1 = \gamma_{x_1,x_2}$, $\beta_2 = \gamma_{x_1,y_2}$, $\beta_3 = \gamma_{y_1,y_2}$. The points are $p_1 = (x_1, x_2, t_{x_2,y_2})$, $r_1 = (x_1, y_2, t_{x_2,y_2})$, $r_2 = (x_1, y_2, t_{x_1,y_1})$, $q_2 = (y_1, y_2, t_{x_1,y_1})$. Notice that the length ℓ_0 of α_0 satisfies $|\ell_0 - (d(p, p_1) + d(r_1, r_2) + d(q_2, q))| \leq 2$. Now the claim follows from Theorem 4.1 and the fact that ℓ_0 and $\rho(p, q)$ differ by a bounded constant.

Now let g be an arbitrary left-invariant Riemannian metric on G. Let c be a one-parameter subgroup of G that is perpendicular to N at e with respect to g. Then the above argument goes through with vertical lines replaced with left cosets of c, S_1 replaced with $(N_1 \times \{0\} \times \{0\})c$, and S_2 replaced with $(\{0\} \times N_2 \times \{0\})c$. □

4.3. **Proof of Theorem 4.1.** Let G, d, $d^{(j)}$ and ρ be as in Subsection 4.1.

Lemma 4.6. *The map $\pi_j : (G, d) \to (S_j, d^{(j)})$ is L-Lipschitz for some $L \geq 1$.*

Proof. This follows from the fact that π_j is a Lie group homomorphism. We shall only prove the case when $j = 1$ since the case for $j = 2$ is similar. It suffices to show that the operator norm of the tangential map $D_p\pi_1 : (T_pG, g) \to (T_{\pi_1(p)}S_1, g^{(1)})$ is independent of the point p.

It is easy to check that the following diagram commutes for every $x \in G$:

$$\begin{array}{ccc} G & \xrightarrow{\pi_1} & S_1 \\ \downarrow{\scriptstyle L_x} & & \downarrow{\scriptstyle L_{\pi_1(x)}} \\ G & \xrightarrow{\pi_1} & S_1 \,. \end{array}$$

This leads to the commuting diagram of tangential maps:

$$\begin{array}{ccc} (T_xG, g) & \xrightarrow{D_x\pi_1} & (T_{\pi_1(x)}S_1, g^{(1)}) \\ \Big\downarrow{\scriptstyle D_xL_{x^{-1}}} & & \Big\downarrow{\scriptstyle D_{\pi_1(x)}L_{\pi_1(x^{-1})}} \\ (T_eG, g) & \xrightarrow{D_e\pi_1} & (T_eS_1, g^{(1)})\,. \end{array}$$

Since the metrics g and $g^{(1)}$ are left invariant, the maps $D_xL_{x^{-1}}$ and $D_{\pi_1(x)}L_{\pi_1(x^{-1})}$ are linear isometries. Now the commuting diagram implies that $D_x\pi_1 : (T_xG, g) \to (T_{\pi_1(x)}S_1, g^{(1)})$ and $D_e\pi_1 : (T_eG, g) \to (T_eS_1, g^{(1)})$ have the same operator norm.

□

Let $p = (x_1, x_2, t)$ and $q = (y_1, y_2, s)$ with $t \leq s$. Let $\beta : [0, l] \to G$ be the arc-length parametrization of a length-minimizing geodesic from q to p. Let

$$h_+ := \max\{h(x) | x \in \mathrm{im}(\beta)\}$$

and

$$h_- := \min\{h(x) | x \in \mathrm{im}(\beta)\}.$$

Set $D_+ := t_{x_1,y_1} - h_+$ and $D_- := h_- - t_{x_2,y_2}$.

Lemma 4.7. *There is a constant $C_0 \geq 0$ independent of the points p, q such that $\max\{D_+, D_-\} \leq C_0$.*

Proof of Theorem 4.1 assuming Lemma 4.7. We use the fact that the minimal distance between $N_1 \times N_2 \times \{t\}$ and $N_1 \times N_2 \times \{t'\}$ is $|t - t'|$. Since ℓ_0 is the length of a curve between q and p, we have $d(p, q) \leq \ell_0$. We shall show that the reverse inequality holds up to an additive constant by using the expression for $\tilde{\rho}$. Here we only write down the details for the case $t_{x_2,y_2} \leq \min\{t, s\}$, $t_{x_1,y_1} \geq \max\{t, s\}$ as the other cases are similar. First assume β reaches height h_+ before it reaches height h_-. From q the curve β first reaches the height h_+, so this subcurve has length at least $h_+ - h(q)$. Then β goes down to the height h_-, so the length of this portion of β is at least $h_+ - h_-$. Finally β goes up and reaches the height $h(p)$, so the length of this portion of β is at least $h(p) - h_-$. Hence the length of β is at least

$$\begin{aligned} &(h_+ - h(q)) + (h_+ - h_-) + (h(p) - h_-) \\ &= 2h_+ - 2h_- - (h(q) - h(p)) \\ &= 2t_{x_1,y_1} - 2D_+ - 2t_{x_2,y_2} - 2D_- - (h(q) - h(p)) \\ &= \tilde{\rho}(p, q) - 2 - 2D_+ - 2D_- \\ &\geq \ell_0 - 4 - 2D_+ - 2D_-. \end{aligned}$$

If β reaches height h_- before it reaches height h_+, then a similar argument shows that its length is greater than the quantity above. The theorem now follows from Lemma 4.7. □

Notice that Lemma 4.7 was used only in the last step of the proof above.

Before we proceed with the proof of Lemma 4.7, we first give a rough idea of the arguments. With notation β, D_+ and D_- as above. We may assume $D_+ \geq D_-$. We already observed that the height change of β is at least $\ell_0 - 4D_+ - 4$. The key of the proof is to show that β has a

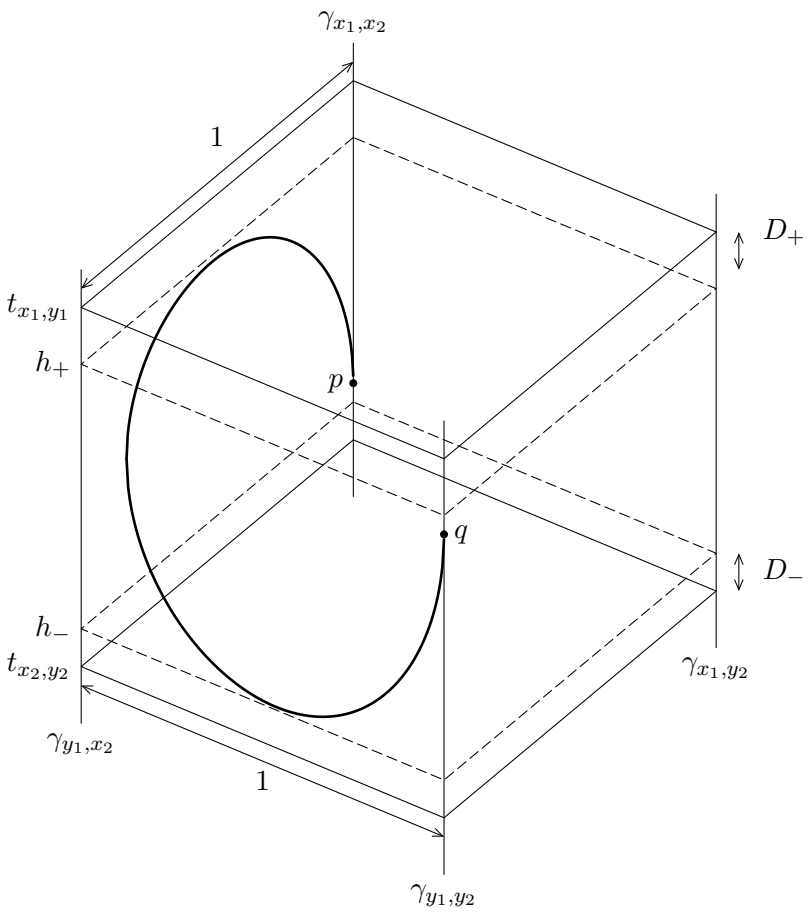

FIGURE 10. Geodesic segment β between p and q in coordinate view.

sub-curve $\tilde{\beta}$ whose length is at least $c_1 2^{c_2 D_+} - c_3 - c_4 D_+$ for suitable constants $c_i > 0$ and whose height change is at most $2D_+$. It then follows that

$$\ell_0 \geq \text{length}(\beta) \geq (\ell_0 - 4D_+ - 4) - 2D_+ + (c_1 2^{c_2 D_+} - c_3 - c_4 D_+),$$

implying an upper bound for D_+ and finishing the proof of Lemma 4.7. To show the existence of such a sub-curve we use Lemma 2.3. We show that there is a sub-curve α (with endpoints $\tilde{p}$ and $\tilde{q}$) of β and vertical geodesics $\tilde{p}\hat{p}$, $\tilde{q}\hat{q}$ (with $\hat{p}$ above $\tilde{p}$ and $\hat{q}$ above $\tilde{q}$ and $h(\hat{p}) = h(\hat{q})$, including the possibility that $\hat{p} = \tilde{p}$ or $\hat{q} = \tilde{q}$) such that $\alpha_1 = \tilde{p}\hat{p} \cup \alpha \cup \tilde{q}\hat{q}$ has the following properties: (1) $d^{(1)}(\pi_1(\hat{p}), \pi_1(\hat{q}))$ is comparable with D_+; (2) $\pi_1 \circ \alpha_1$ satisfies the assumption of Lemma 2.3. By Lemma 2.3 α_1 has a sub-curve α_2 such that the length of $\pi_1 \circ \alpha_2$ is exponential in D_+ and whose height change is comparable with D_+. It is clear that $\tilde{\beta} := \alpha \cap \alpha_2$ has the same properties. Since the map π_1 is Lipschitz, we see that the length of $\tilde{\beta}$ is exponential in D_+ and whose height change is comparable with D_+.

Proof of Lemma 4.7. We will only consider the case $D_+ \geq D_-$ and show that $D_+ \leq C_0$. The case $D_- \geq D_+$ can be similarly handled by considering π_2 instead of π_1. We may assume that

$$D_+ \geq 20 \max\{C, 1\}, \tag{4.3}$$

with C the constant from (4.2), otherwise we are done.

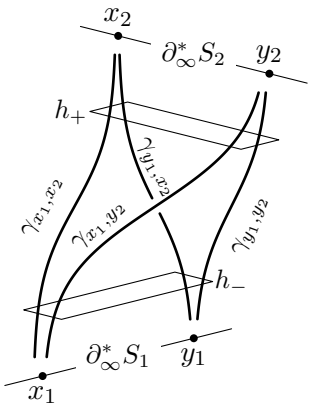

FIGURE 11. Metric view of the four vertical geodesics involved.

We consider three cases depending on the value of $h_+ - D_+$.

Case I: $h_+ - D_+ > h(q)$.

We will divide the curve β into several subcurves. Let $t_1 \in [0,l]$ be the first t such that $h(\beta(t)) \geq h_+ - D_+$. Let $t_2 \in [0,l]$ be the last t such that $h(\beta(t)) \geq h_+ - D_+$. Also let $l_0 \in [0,l]$ be such that $h(\beta(l_0)) = h_-$. (There may be more than one such l_0; we just pick one.)

(1) Subcurve β_1: Set $\beta_1 := \beta|_{[0,t_1]}$. Set $\ell_1 := h_+ - D_+ - h(q)$ if $l_0 \notin [0,t_1]$ and $\ell_1 := (h(q) - h_-) + (h_+ - D_+ - h_-)$ if $l_0 \in [0,t_1]$. By considering the height change we see that $\ell(\beta_1) \geq \ell_1$. Remember that $h(p) \leq h(q)$ by assumption. Write $\beta(t_1) = (a_1, a_2, h_+ - D_+)$ with $a_i \in N_i$.

(2) Subcurve β_2: Set $\beta_2 := \beta|_{[t_1,t_2]}$. Set $\ell_2 := 2D_+$ if $l_0 \notin [t_1,t_2]$ and $\ell_2 := 2(h_+ - h_-)$ if $l_0 \in [t_1,t_2]$ Again, by considering the height change we obtain $\ell(\beta_2) \geq \ell_2$. Write $\beta(t_2) = (b_1, b_2, h_+ - D_+)$ with $b_i \in N_i$.

(3) Subcurve β_3: Set $\beta_3 := \beta|_{[t_2,l]}$. Set $\ell_3 := (h_+ - D_+) - h(p)$ if $l_0 \notin [t_2,l]$ and $\ell_3 := ((h_+ - D_+) - h_-) + (h(p) - h_-)$ if $l_0 \in [t_2,l]$. As above we have $\ell(\beta_3) \geq \ell_3$.

We observe that $\sum \ell_j \geq 2(h_+ - h_-) - (h(q) - h(p)) \geq \ell_0 - 4 - 2D_+ - 2D_- \geq \ell_0 - 4 - 4D_+$. (Recall that ℓ_0 is the length of α_0.)

We now claim that

$$d^{(1)}((x_1, h_+ - D_+), (y_1, h_+ - D_+)) \geq \frac{39}{10} D_+. \tag{4.4}$$

Indeed, by the triangle inequality and the fact that any $d^{(1)}$-geodesic $\gamma_\cap$ between $(x_1, h_+ - D_+)$ and $(y_1, h_+ - D_+)$ lies in a C-neighborhood of the path $\gamma_\sqcap$ defined as a concatenation of vertical geodesics and length-minimizing segment in the horosphere of height t_{x_1,y_1} between the same points (See Figure 13), we have that $d^{(1)}((x_1, h_+ - D_+), (y_1, h_+ -$

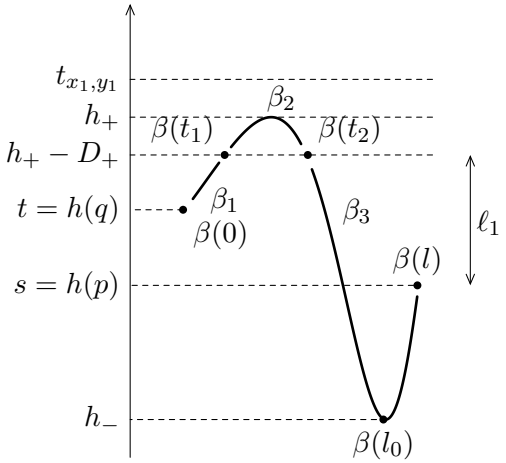

FIGURE 12. Case I with $l_0 \notin [0, t_2]$, image of β through the projection π_1.

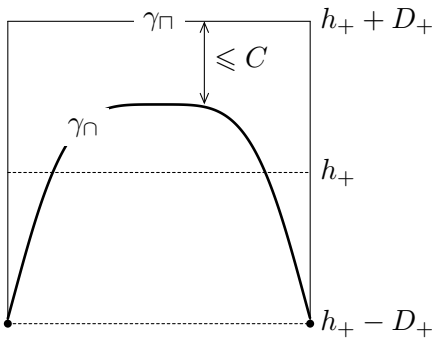

FIGURE 13. Proof of inequality (4.4). The picture is in $(S_1, d^{(1)})$.

$D_+)) \geq 2(2D_+ - C)$ so that

$$d^{(1)}((x_1, h_+ - D_+), (y_1, h_+ - D_+)) \geq 2\left(2D_+ - \frac{1}{20}D_+\right) \qquad \text{by (4.3)}$$
$$= 4D_+ - \frac{1}{10}D_+ = \frac{39}{10}D_+.$$

Since

$$\frac{39}{10} = \frac{7}{2} + \frac{1}{5} + \frac{1}{5},$$

by the triangle inequality one of the following holds (recall that a_1, a_2, b_1, b_2 were introduced under (1) Subcurve β_1 and (2) Subcurve β_2):

(Condition I.1) $$d^{(1)}((y_1, h_+ - D_+), (a_1, h_+ - D_+)) \geq D_+/5;$$

(Condition I.2) $$d^{(1)}((a_1, h_+ - D_+), (b_1, h_+ - D_+)) \geq \frac{7}{2} D_+;$$

(Condition I.3) $$d^{(1)}((b_1, h_+ - D_+), (x_1, h_+ - D_+)) \geq D_+/5.$$

Since β is a minimizing geodesic between q and p and ℓ_0 is the length of a path between p and q we have $\ell(\beta) \leq \ell_0$. On the other hand, Condition (I.1) or (I.2) or (I.3) holds.

We first assume (Condition I.1) holds. Let y be the point on the vertical geodesic γ_{y_1,y_2} through q with $h(y) = h_+ - D_+$, and qy the segment of γ_{y_1,y_2} between q and y. Then the curve $\alpha_1 := \beta_1 \cup qy$ is a path joining y and $\beta(t_1)$ that lies below the horosphere $h = h_+ - D_+$. (Recall that β starts from q.)

To simplify notation denote $q' := \pi_1(y) = (y_1, h_+ - D_+)$ and $p' := \pi_1(\beta(t_1)) = (a_1, h_+ - D_+)$. Then (Condition I.1) simply says $d^{(1)}(p', q') \geq D_+/5$.

Subcase I.1.a: $H(\alpha_1) \leq d^{(1)}(p', q')$. We use Lemma 2.2 to conclude $\ell(\pi_1 \circ \alpha_1) \geq 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C$. By Lemma 4.6 the map π_1 is L-Lipschitz for some $L > 0$, and so we have

$$\ell(\alpha_1) \geq \ell(\pi_1 \circ \alpha_1)/L \geq \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L.$$

As $d(q, y) \leq H(\alpha_1) \leq d^{(1)}(p', q')$, we have $\ell(\beta_1) = \ell(\alpha_1) - d(q, y) \geq \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - d^{(1)}(p', q')$. Note that $\ell_1 \leq 2H(\beta_1) = 2H(\alpha_1) \leq 2d^{(1)}(p', q')$. Now we have

$$\begin{aligned}
\ell_0 &\geq \ell(\beta) \\
&= \ell(\beta_1) + \ell(\beta_2) + \ell(\beta_3) \\
&\geq \ell(\beta_1) + \ell_2 + \ell_3 \\
&= (\ell(\beta_1) - \ell_1) + (\ell_1 + \ell_2 + \ell_3) \\
&\geq (\frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - d^{(1)}(p', q') - 2d^{(1)}(p', q')) + (\ell_0 - 4 - 4D_+) \\
&\geq \ell_0 - 4 - 23 d^{(1)}(p', q') + \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L,
\end{aligned}$$

where for the last inequality we used $d^{(1)}(p', q') \geq D_+/5$. Now it is clear that $d^{(1)}(p', q')$ and so D_+ is bounded above by a constant depending only on L, C, and δ.

Subcase I.1.b: $H(\alpha_1) > d^{(1)}(p', q')$. Then Lemma 2.3 (2) applied to $\pi_1 \circ \alpha_1$ implies there is a subcurve $\tilde{\beta}_1$ of β_1 that is either a segment or the union of two segments of β_1 such that the height change (of the end points of the segments in $\tilde{\beta}_1$) is at most $4d^{(1)}(p', q')$ and the length of $\pi_1 \circ \tilde{\beta}_1$ satisfies

$$\ell(\pi_1 \circ \tilde{\beta}_1) \geq 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C - 3d^{(1)}(p', q').$$

On the other hand, the map π_1 is L-Lipschitz so we have $\ell(\tilde{\beta}_1) \geq \ell(\pi_1 \circ \tilde{\beta}_1)/L$. Let $\tilde{\tilde{\beta}}_1$ be the complement of $\tilde{\beta}_1$ in β_1. Since the height change of β_1 is at least ℓ_1 and the height change of $\tilde{\beta}_1$ is at most $4d^{(1)}(p', q')$, we see the length of $\tilde{\tilde{\beta}}_1$ satisfies

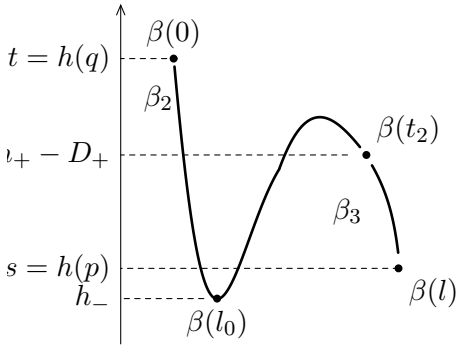

FIGURE 14. π_1-projection of β in Case II when $l_0 \in [0, t_2]$.

$\ell(\tilde{\tilde{\beta}}_1) \geq \ell_1 - 4d^{(1)}(p', q')$. Together with the estimate of the length of $\tilde{\beta}_1$ from the above paragraph we get $\ell(\beta_1) \geq \ell_1 + \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - (3/L + 4)d^{(1)}(p', q')$. It follows that

$$\begin{aligned}
\ell_0 &\geq \ell(\beta) \\
&\geq \ell_1 + \ell_2 + \ell_3 + \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - (3/L + 4)d^{(1)}(p', q') \\
&\geq \ell_0 - 4 - 4D_+ + \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - (3/L + 4)d^{(1)}(p', q') \\
&\geq \ell_0 - 4 + \frac{1}{L} 2^{\frac{d^{(1)}(p',q')-C-2}{2\delta}} - C/L - (3/L + 24)d^{(1)}(p', q'),
\end{aligned}$$

where again for the last inequality we used $d^{(1)}(p', q') \geq D_+/5$. Now it is clear that $d^{(1)}(p', q')$ and so D_+ is bounded above by a constant depending only on L, C, and δ.

This finishes the proof of Claim (4.4) in Case I, Condition (Condition I.1).

Now assume (Condition I.2) holds. This condition implies $d^{(1)}((a_1, h_+), (b_1, h_+)) > D_+$. Since $\alpha_2 := \gamma_{a_1,a_2}|_{[h_+-D_+,h_+]} \cup \beta_2 \cup \gamma_{b_1,b_2}|_{[h_+-D_+,h_+]}$ lies below the height h_+, an argument similar to the case of (Condition I.1) finishes the proof.

Finally we assume (Condition I.3) holds. In this case the curve $\alpha_3 := \gamma_{x_1,x_2}|_{[h(p),h_+-D_+]} \cup \beta_3$ lies below the height $h_+ - D_+$ and we repeat the above argument.

Case II: $h(p) < h_+ - D_+ \leq h(q)$.

In this case we divide the curve β into two subcurves. Let $t_2 \in [0, l]$ be the last t such that $h(\beta(t)) \geq h_+ - D_+$. Also let $l_0 \in [0, l]$ be such that $h(\beta(l_0)) = h_-$.

(1) Subcurve β_2: Set $\beta_2 := \beta|_{[0,t_2]}$. Set $\ell_2 := (h_+ - h(q)) + (h_+ - (h_+ - D_+))$ if $l_0 \notin [0, t_2]$ and $\ell_2 := (h_+ - h(q)) + (h_+ - h_-) + (h_+ - D_+ - h_-)$ if $l_0 \in [0, t_2]$. By considering the height change we obtain $\ell(\beta_2) \geq \ell_2$. Write $\beta(t_2) = (b_1, b_2, h_+ - D_+)$ with $b_i \in N_i$.

(2) Subcurve β_3: Set $\beta_3 := \beta|_{[t_2,l]}$. Set $\ell_3 := (h_+ - D_+) - h(p)$ if $l_0 \notin [t_2, l]$ and $\ell_3 := ((h_+ - D_+) - h_-) + (h(p) - h_-)$ if $l_0 \in [t_2, l]$. As above we have $\ell(\beta_3) \geq \ell_3$.

We observe that $\ell_2 + \ell_3 \geq 2(h_+ - h_-) - (h(q) - h(p)) \geq \ell_0 - 4 - 4D_+$ as before.

As before $d^{(1)}((x_1, h_+ - D_+), (y_1, h_+ - D_+)) \geq \frac{39}{10} D_+$ and so by the triangle inequality one of the following holds:

(Condition II.1) $$d^{(1)}((y_1, h_+ - D_+), (b_1, h_+ - D_+)) \geq \frac{37}{10} D_+;$$

(Condition II.2) $$d^{(1)}((b_1, h_+ - D_+), (x_1, h_+ - D_+)) \geq D_+/5.$$

First assume (Condition II.1) holds. This implies $d^{(1)}((y_1, h_+), (b_1, h_+)) > D_+$. The curve $\alpha_2 := \gamma_{b_1,b_2}|_{[h_+ - D_+, h_+]} \cup \beta_2 \cup \gamma_{y_1,y_2}|_{[h(q), h_+]}$ lies below the height h_+ and we can repeat the argument in Case I to finish the proof.

Finally we assume (Condition II.2) holds. In this case the curve $\alpha_3 := \gamma_{x_1,x_2}|_{[h(p), h_+ - D_+]} \cup \beta_3$ lies below the height $h_+ - D_+$ and we can repeat the argument in Case I to finish the proof.

Case III: $h_+ - D_+ \leq h(p)$. As before we have $d^{(1)}((x_1, h_+ - D_+), (y_1, h_+ - D_+)) \geq \frac{39}{10} D_+$, which implies $d^{(1)}((x_1, h_+), (y_1, h_+)) > D_+$. In this case the curve $\alpha := \gamma_{x_1,x_2}|_{[h(p), h_+]} \cup \beta \cup \gamma_{y_1,y_2}|_{[h(q), h_+]}$ lies below the height $h = h_+$ and we can apply the previous argument to conclude. □

5. Reformulating former results

5.1. **A special case of the pointed sphere conjecture.** We shall refer here to the pointed sphere conjecture of Cornulier recorded in [Cor18, Conjecture 19.104].

By first stratum of a Carnot algebra with Carnot derivation D, we mean the eigenspace $\ker(D - 1)$, which by assumption Lie generates the Carnot algebra, see [LD17]. The higher strata are the subspaces $\ker(D - i)$ for $i \geqslant 2$; the Lie algebra is a direct sum of its strata. (One also encounters the term *layer* in the literature.)

Let N be a Carnot group with Lie algebra $\mathfrak{n}$ and first stratum V_1. We say that N (or equivalently $\mathfrak{n}$) has *reducible first stratum* if there is a nontrivial subspace W of V_1 such that for every strata-preserving automorphism ϕ of $\mathfrak{n}$ one has $\phi(W) = W$. Such a notion has been studied in [Xie13], however, the reader should not mistake it with the notion of reducibility from the same paper. The group of strata-preserving automorphisms is also called the *group of graded automorphisms*.

Proposition 5.1. *Let $S = N \rtimes \mathbb{R}$ be a Heintze group of Carnot type. Assume that N has reducible first stratum. Then, the pointed sphere conjecture holds for S. Namely, every quasi-symmetric self-homeomorphism of ∂S fixes the focal point in ∂S.*

Proof. The argument that we shall follow is similar to the one in [LDX16] and it is based on [Xie13], such a principle goes back to [Pan89a, before Corollaire 6.9]. Namely, we shall prove that the focal point in ∂S is fixed by proving that a special foliation in ∂S is preserved (See Figure 15).

Let ω be the focal point in ∂S, so that $\partial S = N \cup \{\omega\}$. Let $F \colon \partial S \to \partial S$ be a quasi-symmetric homeomorphism. We need to prove that $F(\omega) = \omega$. Let us assume that this is not the case.

Since N is assumed to have reducible first stratum V_1, then there is a nontrivial subspace W of V_1 that is fixed by (the differential of) every strata-preserving automorphism of N. Consequently, the nontrivial group G generated by $\exp(W)$ is preserved by every strata-preserving automorphism of N.

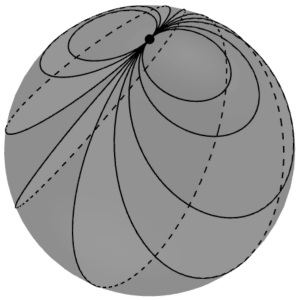

FIGURE 15. Invariant foliation on the boundary at infinity.

The cosets of G induce a singular foliation on ∂S. In particular, when we restrict to the sets $U_1 := N \setminus F^{-1}(\omega)$ and $U_2 := N \setminus F(\omega)$, we have that the leaves on U_1 (resp. on U_2) are exactly the left cosets xG, as $x \in N$, except for a leave, which is $F^{-1}(\omega)G \setminus F^{-1}(\omega)$ (resp. $F(\omega)G \setminus F(\omega)$). At this point we stress that in ∂S while $F^{-1}(\omega)$ and $F(\omega)$ are in the closure of just one of these leaves, the point ω is in the closure of every leave.

Let us restrict to the map $\hat{F} := F|_{U_1} : U_1 \to U_2$. Since U_1 and U_2 are open set of the Carnot group N, by Pansu's differentiability theorem [Pan89b], the map $\hat{F}$ is Pansu differentiable at almost every point and its Pansu differential is a strata-preserving automorphism, which therefore preserves the proper subgroup G. By the argument in [Xie13, Proposition 3.4] we have that $\hat{F}$ preserves the leaves of the foliation that we are considering. Since we had a topological characterization of the point ω (and since the map F is continuous), then we get to a contradiction unless F fixes ω. □

5.2. **Restatement of Le Donne-Xie's theorem.**

Theorem 5.1 (After Le Donne-Xie)**.** *Let N be a Carnot group with reducible first stratum. Let $S = N \rtimes \mathbb{R}$ be the Carnot-type Heintze group associated to N, and equip S with any left-invariant Riemannian distance. Then, every self-quasiisometry of S is a rough isometry.*

Proof. Let g_0 be a left-invariant Riemannian metric on $S = N \rtimes \mathbb{R}$ such that the N direction is perpendicular to $\mathbb{R}$, let d_0 be the associated distance. Then the ideal boundary ∂S can be identified with $N \cup \{\omega\}$, and the Carnot metric on N is a parabolic visual metric with respect to ω. Now let g be an arbitrary left-invariant Riemannian metric on S with d the associated distance, and let ϕ be a self quasiisometry of (S, d). Then ϕ is also a self quasiisometry of (S, d_0). By Proposition 5.1, the map $\partial\phi : \partial S \to \partial S$ fixes the focal point of S and so induces a self quasisymmetric map $\partial^*\phi$ of N (with the Carnot metric). Then $\partial^*\phi$ is a biLipschitz homeomorphism of N [LDX16, Theorem 1.2]. However, a self quasiisometry of a Gromov-hyperbolic space is a rough isometry if and only if the induced boundary map is biLipschitz; this follows from the results of Bonk-Schramm [BS11] (Theorems 7.4 and 8.2) or, for a direct proof in the case of parabolic visual metric, see [SX12, Lemma 5.1]. Hence $\phi : (S, d_0) \to (S, d_0)$ is a rough isometry. By Theorem A the identity map $\mathrm{Id} : (S, d_0) \to (S, d)$ is a rough isometry. It follows that $\phi : (S, d) \to (S, d)$ is also a rough isometry. □

Remark 5.2. Pansu defined the *Carnot-type groups of class* (C) in [Pan89b, 14.1]. At the Lie algebra level, the definition reads as follow: the Carnot-type group $N \rtimes_D \mathbb{R}$ (where N is

different from $\mathbb{R}$) is of Pansu's class (C) if the centralizer of $\mathbb{R}D$ in the Lie algebra of derivations of $\mathfrak{n}$ is equal to $\mathbb{R}D$ itself. Since the centralizer of $\mathbb{R}D$ Lie generates the strata-preserving automorphisms of $\mathfrak{n}$, and since the first stratum V_1 of a nilpotent Lie algebra has dimension greater than or equal 2, the first stratum of a Lie algebra of class (C) has a vector w_1 generating a proper subspace that is invariant under the strata-preserving automorphisms (in fact, every nonzero vector in the first stratum is good for this). It follows that the groups in Pansu's class (C) have reducible first stratum. Pansu proved that among Carnot groups N of class 2 with $\dim V_1$ even, greater or equal to 10 and $3 \leqslant \dim N - \dim V_1 \leqslant 2\dim V_1 - 3$, the property of being of class (C) is *generic* in the sense of algebraic geometry [Pan89b]. This implies that having reducible first stratum is also a generic property among these groups. As mentioned in the Introduction, Theorem 5.1 for the groups in Pansu's class (C) is due to Pansu.

5.3. **Restatement of Eskin-Fisher-Whyte's and Ferragut's theorems.**

Theorem 5.3 (After Eskin-Fisher-Whyte)**.** *Let G be the Lie group SOL. Equip G with any left-invariant Riemannian distance. Then, every self-quasiisometry of G is a rough isometry.*

Theorem 5.4 (After Ferragut)**.** *Let G be a non-unimodular Sol-type group. Equip G with any left-invariant Riemannian distance. Then, every self-quasiisometry of G is a rough isometry.*

Proof of Theorems 5.3 and 5.4. Start assuming that G is a non-unimodular Sol-type group. Equip G with a horospherical product Riemannian metric g_0, that is, a metric for which $\mathfrak{n}_1 \perp \mathfrak{n}_2$. Decompose $G = N_1 \times N_2 \times \mathbb{R}$ where the direction of the $\mathbb{R}$ factor is g_0-perpendicular to $N_1 \times N_2$. By [Fer22, Theorem 10.3.2], every quasiisometry Φ of G is a bounded distance away from $(\Psi_1, \Psi_2, \mathrm{Id}_{\mathbb{R}})$, where Ψ_i is bilipschitz with respect to the D_i-parabolic metric on N_i. Using [SX12] we have that the map $(\Psi_1, \mathrm{Id}_{\mathbb{R}})$ is a rough isometry of S_1 while $(\Psi_2, \mathrm{Id}_{\mathbb{R}})$ is a rough isometry of S_2. And then we conclude by Theorem 4.1 that Φ is a rough isometry of (G, g_0). Then by Theorem B, Φ is a rough isometry of G with respect to every left-invariant Riemannian distance. Now, let G be the three-dimensional Lie group SOL equipped with its standard metric written in coordinates (n_1, n_2, t) as

$$ds^2 = e^{-2t}dn_1^2 + e^{2t}dn_2^2 + dt^2,$$

and Φ is a quasiisometry of G, it follows from [EFW13] that up to possibly composing Φ with an isometry, Φ is at bounded distance from a product map of the form above. The end of the argument is the same as before. □

5.4. **Restatement of Carrasco Piaggio's theorem.**

Theorem 5.5 (After Carrasco Piaggio)**.** *Let S be a Heintze group. Assume that the real shadow of S is not of Carnot type. Equip S with a left-invariant Riemannian metric. Then, every self quasiisometry of S is a rough isometry.*

Proof. Let g_0 be the left-invariant Riemannian metric on S which simultaneously is isometric to a left-invariant metric $\tilde{g}_0$ on the real shadow S_0 [Ale75]; denote by $\rho\colon S \to S_0$ any such isometry. By the published version of [CP17, Corollary 1.8], every self quasiisometry of S_0 is a rough isometry. Let ϕ be a self quasiisometry of S. Then $\rho\phi\rho^{-1}$ is a self quasiisometry of S_0, hence a rough isometry of S_0 with respect to $\tilde{g}_0$. It follows that ϕ is a rough isometry of g_0, and then of every left-invariant metric by Theorem A. □

5.5. **Restatement of Kleiner-Müller-Xie's theorems.** Recall that a Carnot group N is *rigid* in the sense of Ottazzi-Warhurst if for every open subset $U \subset N$, the space of contact maps $f : U \to N$ is finite dimensional, and *nonrigid* otherwise. A smooth map $f : U \to N$ is *contact* if its differential preserves the first stratum.

Theorem 5.6 (After Kleiner, Müller and Xie)**.** *Let S be a Heintze group whose real shadow is of Carnot type. Assume that the nilradical $N = [S, S]$ is nonrigid in the sense of Ottazzi-Warhurst, and that N is not $\mathbb{R}^d$ nor a Heisenberg group. Equip S with a left-invariant Riemannian metric. Then, every self quasiisometry of S is a rough isometry.*

Proof. The pointed sphere conjecture holds for these groups by [KMX21, Theorem 1.2] and global quasisymmetric homeomorphisms of the boundary minus the focal point are bilipschitz by [KMX21, Theorem 3.1]. The mechanism of proof is then exactly the same as for Theorem 5.1. □

Theorem 5.7 (After Kleiner, Müller and Xie)**.** *Let S be a Heintze group whose real shadow is of Carnot type. Assume that the nilradical $N = [S, S]$ is the group of unipotent triangular real $n \times n$ matrices, $n \geqslant 4$. Equip S with a left-invariant Riemannian metric. Then, every self quasiisometry of S is a rough isometry.*

Proof. Global quasisymmetric homeomorphisms of the boundary minus the focal point are bilipschitz by [KMX22, Theorem 1.3]. By [KMX22, Corollary 3.2], there is an automorphism τ of N, such that possibly after composing with τ, every local quasiconformal homeomorphism of the boundary locally preserves a coset foliation. The pointed sphere conjecture for S can be deduced in the same way as we did in §5.1. □

6. Limitations of the present work and questions left open

6.1. **Failure of the analogous property for Lamplighter groups.** The groups $L_m = \mathbb{Z}/m\mathbb{Z} \wr \mathbb{Z}$ for $m \geqslant 2$ share their asymptotic cones (namely, horospherical products of two $\mathbb{R}$-trees equipped with a preferred horofunction, see [Cor08, Section 9]) with that of the group SOL, and their large-scale geometries are in many respect comparable. However, we will prove below that they fail to have their word metrics roughly similar through the identity map.

Consider the following infinite presentation of the group L_m:

$$\langle a, t \mid a^m, [t^i a t^{-i}, t^j a t^{-j}], i, j \in \mathbb{Z} \rangle$$

and the two finite generating sets

- The wreath product generating set $S_w = \{a, t\}$
- The automaton generating set $S_a = \{t, ta\}$.

We denote by d_w and d_a the word distances with respect to $S_w \cup S_w^{-1}$ and $S_a \cup S_a^{-1}$ respectively. In the following, by "color" we mean an element of $\mathbb{Z}/m\mathbb{Z}$. An element of L_m is encoded by a lighting function $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ together with the position of a cursor, with the following multiplication law: Multiplying by t on the right amounts to moving the cursor to the right, and multiplying by a on the right amounts to shifting color at the position of the cursor. Let n be a positive integer (think of it large enough). Let us first consider the element $g \in L_m$ for which the cursor is located at $0 \in \mathbb{Z}$ and the bulb at position n is lit with the color $1 \in \mathbb{Z}/m\mathbb{Z}$ (all the others being not lit). Note that $g = t^n a t^{-n}$. Since in both generating sets, the cursor

moves at most by one unit at each multiplication by a generator, the distance from 1 to g in both word metrics must be at least $2n$. In fact, one computes that

$$d_w(1, t^n a t^{-n}) = 2n + 1$$

while

$$t^n a t^{-n} = (ta)^n a (ta)^{-n} = (ta)^n t^{-1}(ta)(ta)^{-n} = (ta)^n t^{-1}(ta)^{-(n-1)}$$

hence $d_a(1, t^n a t^{-n}) = 2n$. It follows that if d_a and d_w were to be roughly similar through the identity, they should differ by a constant. However, $d_w(1, (ta)^n) = 2n$ while $d_a(1, (ta)^n) = n$ as may be proved by counting the occurrences of t and the number of bulbs lit in the final configuration.

Remark 6.1. The wreath product metric is easier to understand, and there are explicit formulae for the word length for families of words in normal form. Taback and Cleary have investigated the geometry of the automata metric and the result above could be deduced from their paper [CT05].

6.2. **In search of a coarse notion.** One of the main limitations of our present work is that the property that we identify, namely having all the left-invariant Riemannian metrics roughly similar, is not a coarse property. Indeed, Riemannian metrics play no special role among proper geodesic metrics as far as large-scale geometry is concerned.

The search for a coarse notion leads to the following considerations. Let G be a locally compact, compactly generated group. Denote by $\mathrm{Geom}(G)$ the collection of geometric actions of G, that is, pairs (X, α) where X is proper geodesic and $\alpha\colon G \to \mathrm{Isom}(X)$ is continuous, proper and cocompact. For every pair $\{(X, \alpha), (Y, \beta)\}$ in $\mathrm{Geom}(G)$, and for every pair of points $o_X \in X, o_Y \in Y$, the map $G.{}_\alpha o_X \to G.{}_\beta o_Y$ determined by the identity map of G is a quasiisometry $X \to Y$. We call this map (to be considered only up to bounded distance) the G-orbital map. If $H < G$ is closed and co-compact, then H is still compactly generated locally compact [CdlH16, 2.C.8(3)], and there is a natural map $\mathrm{Geom}(G) \to \mathrm{Geom}(H)$ obtained by $(X, \alpha) \to (X, \alpha_{|H})$. If $K < G$ is a compact normal subgroup, and $\pi : G \to G/K$ is the associated epimorphism, then there is a natural map $\mathrm{Geom}(G/K) \to \mathrm{Geom}(G)$ obtained by $(X, \alpha) \to (X, \alpha \circ \pi)$. We essentially proved the following.

Proposition 6.1. *Let H be a compactly generated locally compact group. Assume that for every pair $\{(X, \alpha), (Y, \beta)\}$ in $\mathrm{Geom}(H)$, the orbital map $X \to Y$ is a rough similarity. Then*

(1) If K is a compact normal subgroup of H, then for every pair $\{(X, \alpha), (Y, \beta)\}$ in $\mathrm{Geom}(H/K)$, the orbital map $X \to Y$ is a rough similarity.

(2) If there exists an injective homomorphism with closed and co-compact image from H to G, then for every pair $\{(X, \alpha), (Y, \beta)\}$ in $\mathrm{Geom}(G)$, the orbital map $X \to Y$ is a rough similarity.

Note that two-ended groups have the property in the proposition. Also, if $\Gamma = H$ is a finitely generated group which sits as a uniform lattice in a locally compact group G, the property expressed by Proposition 6.1 transfers from Γ to G, but not from G to Γ.

6.3. **Final questions.** An affirmative answer to the next question would provide a robust generalization of the main results of the present paper.

Question 6.2. Let G be a completely[4] solvable Lie group with $H^1(G,\mathbb{R}) = \mathbb{R}$. Does it hold that for every pair $\{(X,\alpha),(Y,\beta)\}$ in $\mathrm{Geom}(G)$, the orbital map $X \to Y$ is a rough similarity?

Question 6.3. Same question as above, where $G = \mathbb{Z}/m\mathbb{Z}((t))^2 \rtimes \mathbb{Z}$ is the locally compact group that contains L_m as a lattice (See e.g. [Tes16, Section 2]).

Question 6.4. Same question as above, where $G = (N \times \mathbb{Q}_m) \rtimes \mathbb{Z}$, where N is a nilpotent connected Lie group and $\mathbb{Z}$ acts by multiplication by m on $\mathbb{Q}_m$ and by a contracting automorphism on N.

If the answer to Question 6.4 is yes, then the main theorem of [Dym14] would enter the framework of Theorem C.

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