CARNOT-CARATHÉODORY METRICS AND QUASIISOMETRIES OF RANK ONE SYMMETRIC SPACES

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Abstract (Original abstract in English)

We exhibit a rigidity property of the simple groups $\mathrm{Sp}(n,1)$ and $F_4^{(-20)}$ which implies Mostow rigidity. This property does not extend to $\mathrm{O}(n,1)$ and $\mathrm{U}(n,1)$. The proof relies on quasiconformal theory applied in the CR setting. Extensions are given to a class of solvable Lie groups. As a byproduct of the proof, a result on quasiisometries of infinite nilpotent groups is obtained.

Contents

This is the translation of P. Pansu's "Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un", *Ann. Math*, **129** (1989), 1–60, originally published in French. Notes are included in the end to add context or indicate subsequent developments. Original page numbers are in the margin.

In this paper we establish a rigidity property of the rank one simple Lie groups $\mathrm{Sp}(n,1)$, $n \geqslant 2$ and F_4^{-20}, which implies Mostow rigidity:

Theorem 1. *Every quasiisometry of quaternionic hyperbolic space* $\mathbf{H}\mathrm{H}^n$*, $n \geqslant 2$, (resp. of the Cayley hyperbolic plane* $\mathbf{Ca}\mathrm{H}^2$*) lies a bounded distance away from an isometry, i.e. it differs from an isometry by an application which moves points a bounded distance away.*

A map f between metric spaces is a quasiisometry if there exists constants L and C such that the image of f is C-dense and that, for every $x \neq y$,

$$-C + \frac{1}{L} d(x,y) \leqslant d(fx,fy) \leqslant d(x,y) + C.$$

A quasiisometry between G and G' is somehow a virtual isomorphism in the topological category. (Indeed, it corresponds to an action of G on a C^0 principal bundle with group G' and a compact basis, cf. [Wil82].) An isomorphism between co-compact subgroups of Lie groups extends to a quasiisometry between the corresponding symmetric spaces or the Lie groups. If it is a bounded distance away from an isometry of the symmetric spaces (resp. to an isomorphism of the Lie groups) then the subgroups are conjugated: this is Mostow rigidity [Mos73].

The property expressed by the Theorem 1 is not shared by the groups $\mathrm{O}(n,1)$ nor $\mathrm{U}(n,1)$ Indeed (§11.7 below), $\mathrm{O}(n,1)$ and $\mathrm{U}(n,1)$ have many quasiisometries. They have at least as many quasiisometries as there are diffeomorphisms of the sphere S^{n-1} (resp. as many as the contact transformations of the sphere S^{2n-1}). Like Kazhdan's property (see

[Zim84]), our result indicates that a line may be drawn separating the groups of rank one[1].

However, it is a negative result. It shows that one cannot extend to the lattices of $\mathrm{Sp}(n,1)$ and F_4^{-20} the considerations on the quasiconformal deformations of Kleinian groups of [Thu79]. Especially, it indicates that the concept of quasiisometry is not well adapted to the study of simple groups of rank one, and that one should look for weaker conditions.

Nevertheless, quasiisometries are a efficient tool of investigation for general manifolds of negative curvature. Our method easily extends to a class of negatively curved homogeneous spaces, called "Carnot type"[2].

We are able to classify these spaces up to quasiisometry. We found that "generically" they share the property of $\mathrm{Sp}(n,1)$: their only quasiisometries are isometries. However, there exists homogeneous spaces which do not have it. Often, the quasiisometries form a finite dimensional Lie group. Sometimes, they form an infinite dimensional one. Nevertheless, we feel, without being able to express it clearly, that among all the negatively curved manifolds, the real hyperbolic spaces and complex hyperbolic spaces have the most quasiisometries[3].

1. The method

The proof by Mostow of the strong rigidity for $\mathrm{O}(n,1)$ is based on the idea that quasiisometries correspond to quasiconformal homeomorphisms of the "sphere at infinity" S^n, and uses the regularity of theses transformations. Mostow proved that this idea applies to the other rank one symmetric spaces [Mos73].

We show that the conformal geometry of the spheres at infinity is modeled on nilpotent groups equipped with nonRiemannian metrics. For the latter, one can speak of quasiconformal tansformations[4]. In fact, we prove that the only global quasiconformal homeomorphisms of the sphere at infinity are extensions of isometries.

To explain this rigidity, let us show that it is immediate in the case of diffeomorphisms of the sphere at infinity of regularity at least C^3. The sphere at infinity ∂X of $X = \mathbf{K}\mathrm{H}^n$, $n \geq 2$ supports a plane field Q which is invariant under isometries; its codimension is $\dim \mathbf{K} - 1$. A diffeomorphism ϕ of ∂X is quasiconformal if and only if it preseves Q. If this is the case, then ϕ preserves a G-structure, $G \subset \mathrm{Gl}(Q)$, naturally associated to the plane field Q. When $\mathbf{K} = \mathbf{R}$ there is no 3
special structure, $G = \mathrm{Gl}(Q)$. When $\mathbf{K} = \mathbf{C}$ it is a symplectic conformal structure: $G = C\mathrm{Sp}(2n-2, \mathbf{R})$. These two G-structures have infinite type. When[5] $\mathbf{K} = \mathbf{H}$ it is a quaternionic conformal structure, $G =$

$C\mathrm{Sp}(n-1)\mathrm{Sp}(1)$, and when $\mathbf{K} = \mathbf{Ca}$, $G = C\mathrm{Spin}(7)$, where $\mathrm{Spin}(7) \subset SO(8)$. The latter two G-structures have finite type, their group of C^3-automorphisms has finite dimension, cf. [Tan79]: it is exactly $\mathrm{Sp}(n,1)$ or $F_4^{(-20)}$.

We will now describe the tools that allows one to carry over the argument without any regularity assumption.

1.1. **Carnot-Carathéodory metrics.** Let M be a differentiable manifold. Let V be a subbundle of the tangent bundle TM. For $p \in M$, we denote $V^i(p)$ the subspace of TM generated by the iterated brackets of the vector fields tangent to V at p. One says that V is accessible if there exists r such that $V^r(p) = T_pM$ at every point.

A Carnot-Carathéodory metric on M is given by an accessible subbundle of TM, and a metric on the fibers of V.

From these data one constructs, as in the Riemannian case, a distance. One knows how to define the length of the curves that are *horizontal*, that is, tangent to V. One may then define, for any pair of points x and y of M,

$$d(x,y) = \inf\left\{\mathrm{length}(c) \mid \text{ horizontal } c \text{ between } x \text{ and } y\right\}.$$

This distance defines the usual topology on the manifold M (see [Cho39]); however its Hausdorff dimension is in general larger than $\dim M$. Let us assume in addition that all the V^i are subbundles. Then J. Mitchell proved that the Hausdorff dimension equals

$$\sum_{i=1}^{r} \mathrm{rank}\, V^i,$$

see [Mit85].

In [Gro99], Gromov outlines a notion of “tangent cone” for a metric space. He introduces a topology, called the Gromov-Hausdorff topology, on the set of pointed metric spaces. The tangent cone at x of (X,d) is the limit when it exists of the dilated spaces pointed at x when $\varepsilon \to 0$.

In the case of a Riemannian manifold, the tangent cones are Euclidean spaces. Mitchell proved that for a Carnot-Carathéodory metric such that the V^i are subbundles, the tangent cones are Carnot groups [Mit85][6].

1.2. **Definition.** A *Carnot group* is a simply connected nilpotent Lie group N together with a derivation α on the Lie algebra $\mathcal{N}$ of N, with the following property: the subspace $V^1 = \ker(\alpha - 1)$ generates $\mathcal{N}$.

More concretely, denote $V^{i+1} = [V^1, V^i]$. These subspaces form a graduation of N, i.e.,

$$\mathcal{N} = \sum V^i$$
$$[V^i, V^j] \subset V^{i+j}$$

and one has $\alpha = i$ on V^i. Therefore, a Carnot group is the Lie group of a graded Lie algebra, which is generated by its elements of degree one. To any norm on V_1 corresponds a Carnot-Carathéodory metric on N, which is left-invariant, and for which the automorphism $e^{t\alpha}$ is a homothety of ratio e^t.

Carnot groups appear naturally in analysis, see [Goo76]; they form a class of examples worth of interest in control theory, see [Bro82, 1.3].

1.3. **Differentiation.** In our mind, a Carnot group is an immediate generalization of a vector space, with its translations and homotheties (the latter being generated by the identity derivation).

For continuous applications between Carnot groups, we can speak of differentiability. Let $f : N \to N'$ be a continuous application defined on a neighborhood of the origin 0, and such that $f(0) = 0$. We shall say that f is differentiable at the origin if the sequence of applications

$$e^{t\alpha'} f e^{-t\alpha}$$

converges uniformly on any compact set when t goes to infinity. The limit is called the *differential* of f at 0, and denoted $Df(0)$. Using the translations of N and N', we define the differentiability at any point.

When the norm on V_1 is changed, the Carnot-Caratheodory metric remains bilipschitz to itself. We can therefore speak of Lipschitz applications and quasi-conformal transformations between Carnot groups without referring to a particular Carnot-Caratheodory metric. We recall in §6.1 the definition of a quasi-conformal transformation between metric spaces. Our main result is a generalization of the Rademacher-Stepanov theorem [Ste25].

Theorem 2. *Let N and N' be Carnot groups. Any Lipschitz application (resp. any quasiconformal homeomorphism) from an open set of N and* 5
an open set of N' is almost everywhere differentiable.

1.4. **Absolute continuity.** Given a homeomorphism, is its differential almost everywhere an isomorphism? As in the classical setting, this holds for *absolutely continuous* homeomorphisms, i.e., those that preserve the zero Haar measure sets. Lipschitz homeomorphisms are among these. In the Euclidean setting, this is due du F.W. Gehring

[Geh62] and the proof extends without change to the groups associated to other rank one symmetric spaces [Mos73]. However a new proof is required in the general case: see §7.3.

Note that the results above can be extended to manifolds with Carnot Carathéodory distances. We only give the result in the case of groups. The general case does not follow immediately: except in the Riemannian case, a Carnot-Carathéodory distance is not locally bilipschitz to its tangent cone[7].

1.5. **End of the proof.** Here is how, thanks to Theorem 2, one can extend from diffeomorphisms to homeomorphisms the reasoning outlined at the beginning of this section. Let $X = G/K$ be a a symmetric space of rank one, and $G = KAN$ an Iwasawa decomposition of G. A can be seen as a one-parameter group of automorphisms of the nilpotent group N, generated by a derivation α. It was mentioned above that the sphere at infinity $\partial X = G/MAN$ carried an invariant distribution Q. For any Carnot-Carathéodory metric d attached to this field, the tangent cone at any point of $(\partial X, d)$ is the Carnot group (N, α). According to Theorem 2, a quasi-conformal transformation f of ∂X admits a differential Df almost everywhere. This differential is an automorphism of N which commutes with α. When $G = \mathrm{Sp}(n, 1)$, $n \geqslant 2$ (resp. $G = F_4^{-20}$) , the group $\mathrm{Aut}(\alpha, N)$ of automorphisms that commute to α can be identified with $C\,\mathrm{Sp}(n-1)\,\mathrm{Sp}(1)$ (resp. to $C\,\mathrm{Spin}(7)$ by Proposition 10.1); in particular, the quasi-conformal applications are 1-quasiconformal (corollary 11.2), and a global argument due G.D. Mostow [Mos68] allows us to conclude that they come from isometries.

Our method gives a partial result for the quasi-isometries between nilpotent groups that is described below.

1.6. **Quasiisometries of nilpotent groups.** This problem comes from [Gro81a]. On a finitely generated group Γ, one can consider a family of distances d_Γ, called "algebraic distances". Up to quasiisometry, d_Γ only depends on the structure of Γ. Conversely, it is asked wether two quasiisometric discrete groups are isomorphic, or, at least, commensurable (i.e., one contains a finite index subgroup, isomorphic to one finite index subgroup of the other). This is the case when one of the groups is abelian.

When Γ is nilpotent, we know how to associate a nilpotent Lie group to $\Gamma \otimes \mathbf{R}$. (this construction is due to A. Malcev [Mal51]), and the associated graded Lie group $\mathrm{gr}(\Gamma \otimes \mathbf{R})$. The metric space (Γ, d) has a tangent cone at infinity, which is the Carnot group $\mathrm{gr}(\Gamma \otimes \mathbf{R})$ with

Carnot-Carathéodory metrics [Gro81b, Pan83]. A quasi-isometry between discrete groups gives a bilipschitz homeomorphism between the tangent cones at infinity[8].

Theorem 3. *If two finitely generated nilpotent groups* Γ *and* Γ' *are quasiisometric, then the graded nilpotent Lie groups* $\mathrm{gr}(\Gamma\otimes\mathbf{R})$ *and* $\mathrm{gr}(\Gamma'\otimes\mathbf{R})$ *are isomorphic.*

When the dimension is less or equal to[9] 5, this implies that Γ and Γ' are commensurable. This is not true anymore as soon as the dimension is 6.

1.7. **Organization of the paper.** Part A contains the proof of Theorem 2. Theorem 1 is proved in Part B; a sketch of proof is given in Section 8. In part C, we observe that the rigidity of quaternionic hyperbolic space extends to other homogeneous spaces with negative curvature (Theorem 4).

I thank M. Gromov, who directed this work, proposing Theorem 1 as an objective. Thanks to A. Bellaiche, I understood better why the differentials are groups homeomorphisms. Theorem 3 answers a question of A. Katok, and I thank Spatzier who stimulated me thanks to his insistance. I am happy to thank Y. Benoist as well, for his examples of Lie algebras, F. Ducloux for the representation theory of Spin(7), O. Debarre for the automorphisms of varieties and S. Rickman who initiated me to quasiconformal transformations. Finally, I thank D. Sullivan, A. Koranyi and H. Reimann for their interest in my work.

Part A. Differentiability

2. Scheme of the proof of Theorem 2

As in the proof of Rademacher-Stepanov's theorem, one can reduce the proof, by abstract nonsense (Corollary 3.3 below), to the case $\dim N = 1$, i.e., to the differentiability of rectifiable curves. This (Proposition 4.1), results essentially from Lebesgue's classical theorem on the differentiability almost everywhere of nondecreasing functions of one real 7
variable, together with an estimate of the speed at which a horizontal curve in a Carnot group moves away from the plane V_1 (proposition 4.7).

In the case of quasi-conformal homeomorphisms, it must be shown that "almost every" rectifiable curve is sent on a rectifiable curve. This result, due to G.D. Mostow [Mos73] in the particular case of groups at the boundaries of symmetric spaces of rank one, is the objective of §6.

3. Reduction to the one-dimensional setting

3.1. **Definition.** Let X, X' be metric spaces, let f be a map from X to X'. The *local dilatation* of f, denoted Lip_f, is defined as

$$\mathrm{Lip}_f(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}.$$

3.2. **Proposition.** *Let N, N' be Carnot-graded groups. Let f be a map from an open set of N into N', whose local dilatation is almost everywhere finite. Let $\mu, \nu \in N$. Assume that for almost every $x \in N$, the limits*

$$D_\mu f(x) = \lim_{t \to +\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\mu)),$$
$$D_\nu f(x) = \lim_{t \to +\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\nu))$$

exist. Then for every ω under form $\omega = e^{a\alpha}\mu e^{b\alpha}\nu$ and for almost every $x \in N$, the limit

$$D_\omega f(x) = \lim_{t \to +\infty} e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\omega))$$

exists and is equal to $e^{a\alpha'} D_\mu f(x) e^{b\alpha'} D_\nu f(x)$.

Fix Carnot-Carathéodory distances d and d' on N and N'. Denote by $\mathscr{H}^p$ the Haar measure on N. If $D_\mu f(x)$ exists, then for every a, $D_{e^{a\alpha}\mu} f(x)$ exists, so that one can assume $\omega = \mu\nu$ where $d(1, \mu) = 1$.

Following Egoroff and Lusin (See [Rud87]) for all $\tau > 0$ there exists a closed subset $F \subset N$ such that $\mathcal{H}^p(N \setminus F) < \tau$ and

(i) $D_\mu f(x)$ and $D_\nu f(x)$ exist for all $x \in F$,
(ii) $x \mapsto D_\nu f(x)$ is continuous on F,
(iii) $e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\nu))$ converges to $D_\nu f(x)$, uniformly for $x \in F$.

If one knew that for all t, $xe^{t\alpha}\mu \in F$, the proof would be completed. Indeed, $e^{t\alpha'}\left(f(x)^{-1} f(xe^{-t\alpha}\omega)\right)$ can be decomposed as the product

$$e^{t\alpha'}\left(f(x)^{-1} f(xe^{-t\alpha}\omega)\right) = (1)(2)(3)$$

where

(1) is equal to $e^{t\alpha'}\left(f(x)^{-1} f(xe^{-t\alpha}\mu)\right)$, which tends to $D_\mu f(x)$ by (i);
(2) is equal to $e^{t\alpha'}\left(f(xe^{-t\alpha}\mu)^{-1} f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)\right)\left(D_\nu f(xe^{-t\alpha}\mu)\right)^{-1}$ which-converges to $D_\mu f(x)$ by (iii); and
(3) is equal to $D_\nu f(xe^{t\alpha}\mu)$, which tends to $D_\nu f(x)$ by (ii).

It is not true that $xe^{t\alpha}\mu$ stays in F for every $x \in F$. Nevertheless, if x is a $\mathscr{H}^p$ density point of F, that is, if

$$\limsup_{r\to 0} \frac{\mathscr{H}^p[B(x,r)\setminus F]}{\mathscr{H}^p[B(x,r)]} = 0,$$

then there is a point of F that is close to $xe^{t\alpha}\mu$. Let λ be the distance from $xe^{t\alpha}\mu$ to F. As[10] $B(xe^{-t\alpha}\mu,\lambda)\cap F \neq \emptyset$,

$$\frac{\mathscr{H}^p\left(B(x,e^{-t}+\lambda)\setminus F\right)}{\mathscr{H}^p B(x,e^{-t}+\lambda)} \geqslant \frac{\mathscr{H}^p B(xe^{-t\alpha}\mu,\lambda)}{\mathscr{H}^p B(x,e^{-t}+\lambda)} = \left(\frac{\lambda}{e^{-t}}+\lambda\right)^p,$$

hence $e^t\lambda$ goes to 0 when t goes to $+\infty$, i.e., μ' goes to μ when t goes to $+\infty$. One may then write $e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\omega)\right)$ as the product

$$e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\omega)\right) = (1)(2)(3)(4)(5)$$

where

(1) is equal to $e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\mu)\right)$, which tends to $D_\mu f(x)$ by (i);
(2) is equal to $e^{t\alpha'}\left(f(xe^{-t\alpha}\mu)^{-1}f(xe^{-t\alpha}\mu')\right)$, and
(3) is equal to $e^t\alpha'\left(f(xe^{-t\alpha}\mu')^{-1}f(xe^{-t\alpha}\mu' e^{-t\alpha}\nu)\right) D_\nu f(xe^{-t\alpha}\mu')^{-1}$ which both tend to 1 by (iii);
(4) is equal to $D_\nu f(xe^{-t\alpha}\mu')$ and tends to $D_\nu f(x)$ by (ii); 9
(5) is equal to $e^{t\alpha'}\left(f(xe^{-t\alpha}\mu' e^{-t\alpha}\nu)^{-1}f(xe^{-t\alpha}\mu' e^{-t\alpha}\nu)\right)$.

Assume that f is Lipschitz[11] with Lipschitz constant M. Then

$$d'((2),1) = e^t d'(f(xe^{-t\alpha}\mu), f(xe^{-t\alpha}\mu')) \leqslant Me^t d(e^{-t\alpha}\mu, e^{-t\alpha}\mu') = Md(\mu,\mu')$$

tends to 0 as $t\to+\infty$. Moreover

$$\begin{aligned} d'((5),1) &= e^t d'(f(xe^{-t\alpha}\mu' e^{-t\alpha}\nu), f(xe^{-t\alpha}\mu e^{-t\alpha}\nu)) \\ &\leqslant Me^t d(e^{-t\alpha}(\mu\nu), e^{-t\alpha}(\mu'\nu)) \\ &= Md(\mu\nu,\mu'\nu) \end{aligned}$$

also tends to 0 as $t\to+\infty$. This establishes the existence of

$$\lim_{t\to+\infty} e^{t\alpha'}\left(f(x)^{-1}f(xe^{-t\alpha}\omega)\right)$$

at every density point x of F, that is, almost everywhere on F.

If we now assume that the local dilation is only finite almost everywhere, one should restrict F again. Almost every point of N is in one of the sets

$$A_k = \left\{ x\in N : \forall y\in B(x,1/k),\ \frac{d'(f(x),f(y)}{d(x,y)} \leqslant k \right\}.$$

So one may assume that f is Lipschitz on F, if one removes from F a set of small measure.

3.3. **Corollary.** *Let f be an application from an open set of N into N' such that $\mathrm{Lip}_f < +\infty$ almost everywhere. Let $v_1, \ldots, v_k$ generate the Lie algebra $\mathcal{N}$. If for almost every $x \in N$ the curve*

$$s \mapsto f(x \exp(sv_i))$$

is almost everywhere differentable, then f is differentiable almost everywhere, and the differential $\mu \mapsto D_\mu f(x)$ is a group homomorphism.

The only part left to check is that when x is fixed,

$$f_t(\mu) = e^{t\alpha'}(f(x)^{-1} f(xe^{-t\alpha}\mu))$$

converges uniformly in μ to $D_\mu f(x)$. By assumption, there is a proper submersion from[12] $\mathbf{R}^l$ to N, that we still denote μ, defined by

$$\mu(a_1, \ldots, a_l) = \prod_{j=1}^{l} e^{a_j\alpha} \exp(v_{i_j}).$$

Using the proof of Proposition 3.2, when x is fixed, the speed of convergence of $f_t(\mu)$ to $D_\mu f(x)$ only depends on $\sup \|a_j\|$ and on the speed of convergence of $f_t(v_j)$ to $D_{v_j} f(x)$. That yields the required uniform bound.

4. Differentiability of rectifiable curves

We are going to prove the following proposition.

4.1. **Proposition.** *A locally rectifiable curve $c : \mathbf{R} \to N$ is almost everywhere differentiable. If $\dot{c}(t) \in \mathcal{N}$ is the ordinary derivative, then $\dot{c}(t) \in V^1$ and the derivative is expressed as*

$$D_\mu c(t) = \exp(\mu \dot{c}(t)), \quad \mu \in \mathbf{R}.$$

Before going any further, note that one may assume $\mathrm{Lip}_c \leqslant 1$. Indeed, one may reparametrize c by arclength. The change of parametrization is a homeomorphism between intervals of $\mathbf{R}$, and as such it is almost everywhere differentiable. If h is differentiable at s and $c' = c \circ h^{-1}$ is differentiable at $h(s)$, then c is differentiable at s, and

$$Dc(x) = Dc'(h(s)) \circ \dot{h}(s).$$

It may not hold that c' is differentiable at $h(s)$ for almost every s. Nevertheless, c is differentiable at almost every s. Indeed, let

$$E = \{s : c' \text{ is not differentiable at } h(s)\};$$

then $h(E)$ has zero measure, hence for almost every $s \in E$, $h'(s) = 0$. But then $\mathrm{Lip}_c(s) = 0$, that is, c is differentiable at s with constant differential. Let us translate the statement of Proposition 4.1 by expressing c in exponential coordinates. Set $c(0) = 1$ and

$$c(s) = \exp\left(\sigma^1(s) + \cdots + \sigma^r(s)\right)$$

where $\sigma^i(s) \in V^i$. The curve c is differentiable at 0 if for all i the limit

$$\lim_{s \to 0} s^{-i} \sigma^i(s)$$

exists. In fact we will see that for almost every s, the ordinary derivative $\dot{c}(s)$ is tangent to V^1, do if the differential exists,

$$\begin{cases} \lim_{s \to 0} s^{-1} \sigma^1(s) = \dot{c}(0), & \\ \lim_{s \to 0} s^{-i} \sigma^i(s) = 0 & i > 1. \end{cases}$$

4.2. **The case of the Heisenberg group.** In order to illustrate Proposition 4.1, let us study the case of the three-dimensional Heisenberg group. Let X, Y, Z generate the Lie algebra in such a way that 11

$$[X, Z] = [Y, Z] = 0, \ [X, Y] = Z.$$

Let V^1 be the plane spanned by X and Y and equipped with the Euclidean metric which is such that X, Y form an orthonormal basis. The map $(r, \theta, t) \mapsto \exp(re^{i\theta} + tZ)$ can be used to define a coordinate system almost everywhere on N. The invariant 1-form ω calibrating V^1 can be written

$$\omega = dt - \frac{1}{2} r^2 d\theta.$$

Let $c(s)$ be a smooth curve in N, write $c(s) = c(0)\exp(\sigma^1(s) + \sigma^2(s))$. If it has finite Carnot-Carathéodory length, the curve c is tangent to V^1, and the component $\sigma^2(s) = t(s)Z$ is determined by σ^1 :

$$t(s) = \int_0^{\sigma^1(s)} \frac{1}{2} r^2 d\theta.$$

This means that $t(s)$ is the signed area comprised between the plane curve σ^1 and the line segment $[0, \sigma^1(s)]$.

Fix $v \in V^1$ and set

$$\varepsilon = \sup\left\{ \left\| \frac{\sigma^1(u)}{u} - v \right\| : 0 < u \leqslant s \right\}.$$

By definition, for $u \leqslant s$, $\sigma^1(u)$ is contained in the plane sector of axis v with aperture ε and radius $s \,\mathrm{Lip}_c$, hence $t(s)$ is at most the area of

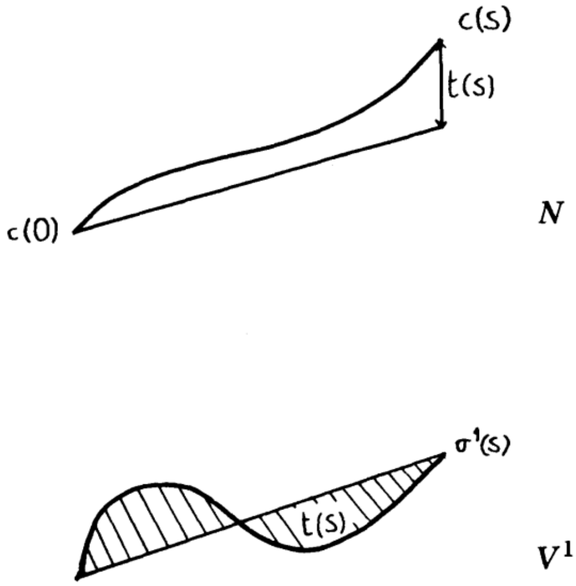

FIGURE 1.

this sector, that is, at most $s^2\varepsilon\,\mathrm{Lip}_c$. Thus

$$\sigma^2(s) = o(s^2)$$

as soon as σ^1 is differentiable at 0. In fact, when we treat the general case of a nonsmooth c, we will need to estimate the total variation of σ^2; in order to achieve this, one needs slightly more than the mere existence of derivative of σ^1 at 0.

Here are some more technical preliminaries useful for the proof of Proposition 4.1. In order to manipulate curves in a Lie group G, we will use the use a developing procedure, which transforms a curve in G into a curve in its Lie algebra, and the inverse procedure: the multiplicative integral.

4.3. **Developing curves.** Let G be a Lie group and let $c : [0,1] \to G$ be a curve such that $c(0) = 1$. To every subdivision $\Sigma = \{0 = t_0 < \cdots < t_N = 1\}$ we can associate a point

$$\sigma_\Sigma = \sum_{k=0}^{N-1} \log(c(t_k)^{-1}c(t_{k+1}))$$

in the Lie algebra $\mathcal{G}$ of G. When the mesh $\|\Sigma\| = \sup|t_{k+1} - t_k|$ tends to 0, this point σ_Σ has a limit. Indeed, if $\Sigma' = \{0 = t'_0 < \cdots < t'_{N'} = 1\}$

is a finer subdivision, where $t_k = t'_l$ and $t_{k+1} = t'_m$, then

$$\left\| \sum_{q=l}^{m-1} \log \left(c(t'_q)^{-1} c(t'_{q+1}) \right) - \log \left(c(t_k)^{-1} c(t_{k+1}) \right) \right\| \leqslant \text{cst.}\, u_k^2$$

where

$$u_k = \sum_{q=l}^{m-1} \left\| \log c(t_q)^{-1} c(t_{q+1}) \right\|$$

(see [Pan83]), so that

$$\|\sigma_{\Sigma'} - \sigma_\Sigma\| \leqslant \sum_{k=0}^{N-1} u_k^2$$

which tends to 0 when $\|\Sigma\|$ tends to 0 if c has finite length. We denote by $\sigma(s)$ the limit of σ_Σ taken with respect to $c_{|[0,s]}$. It is a curve in $\mathcal{G}$ with the same lengthas c. If c is parametrized by arclength, the following estimates hold[13]:

$$(*) \qquad \|\sigma(s) - \log c(s)\| \leqslant \text{cst.}\, s^2$$

$$(**) \qquad \|\sigma(s) - \sigma_\Sigma(s)\| \leqslant \text{cst.}\, s\|\Sigma\|.$$

4.4. **The multiplicative integral.** This is the reverse procedure. Let $\sigma(s)$ be a rectifiable curve in the Lie algebra $\mathcal{G}$, with $\sigma(0) = 0$. One constructs $c(s) = \lim_{\|\Sigma\| \to 0} c_\Sigma(s)$ where Σ is a subdivision of $[0, s]$ and 13

$$c_\Sigma(s) = \prod_{k=0}^{N-1} \exp\left(\sigma(t_{k+1}) - \sigma(t_k)\right).$$

Again, if σ is parametrized by arclength,

$$\|\sigma(s) - \log c(s)\| \leqslant \text{cst.}\, s^2$$

$$\|\log c_\Sigma(s)^{-1} c(s)\| \leqslant \text{cst.}\, s\|\Sigma\| \qquad \text{if } s \leqslant 1.$$

Let us prove that if G is a Carnot group and if c is rectifiable with respect to a Carnot-Carathéodory metric, then σ is in V^1. For this we make use of the following inequality, that is easily check using the dilations $e^{t\alpha}$: for $v \in \mathcal{G}$ such that $v = c_1 + \cdots + v^r$ with $v^i \in V^i$ for all i,

$$(+) \qquad \left\| v^2 + \cdots + v^r \right\| \leqslant \text{cst.}\, d(1, \exp v)^2$$

Assume that c is parametrized by arclength. Set

$$c(s+t) = c(s) \exp\left(\sigma^1(t) + \cdots + \sigma^r(t) \right).$$

Combining (∗) and (+) we obtain that

$$\sigma(s+t)-\sigma(s)=\sigma^1(s)+O(s^2).$$

This proves that the curve σ, which is absolutely continuous, is almost everywhere tangent to V^1. Hence it is contained in V^1. Especially, $\sigma=\sigma^1$ is exactly the projection of c in V^1 identified with $G/[G,G]$.

Let us introduce a last technical tool: the areas swept by curves. Our goal is to prove that, if a curce c is rectifiable with respect to a Carnot-Carathéodory metric, and if the developing curve σ has a derivative at 0, then

$$\lim_{s\to 0} s^{-i}\sigma^i(s)=0$$

for all $i>1$. Using the multiplicative integral, we can approximate each component $\sigma^i(s)$ with a discrete curve σ^i_k, where, a subdivision Σ of $[0,s]$ being given, σ^i_k is

$$\log\prod_{l=0}^{k-1}\exp(\sigma(t_{l+1})-\sigma(t_l))$$

projected onto V^i. The goal is then to estimate $t_k^{-i}\sigma^i_k$. The Campbell-Hausdorff formula gives certain relations that can be used inductively. For instance,

$$\sigma^1_{k+1}-\sigma^1_k=\sigma_{k+1}-\sigma_k,$$
$$\sigma^2_{k+1}-\sigma^2_k=\frac{1}{2}\left[\sigma(t_k),\sigma(t_{k+1})-\sigma(t_k)\right]$$

and so on. We are led to study the convergence of Riemann sums of the form

$$\sum_k\left[\sigma(t_k),\sigma(t_{k+1})-\sigma(t_k)\right].$$

In the case of the three-dimensional Heisenberg group, this sum converges to the area swept by the curve c.

4.5. **Areas swept by a curve in a Lie algebra.** Let $\mathcal{G}$ be a Lie algebra. We denote $[s,y]$ the i-fold Lie bracket $[x,[,\dots,[x,y]\dots]]$ (where x appears $i-1$ times). Let σ be a curve of finite length in $\mathcal{G}$ equipped with a norm. For every subdivision $\Sigma=\{0=t_0<\cdots<t_N=s\}$, set

$$A^i_\Sigma=\sum_{k=0}^{N-1}\left[\sigma(t_k),\sigma(t_{k+1})-\sigma(t_k)\right]_i.$$

Let us prove that when $\|\Sigma\|$ tends to 0, the sequence of points A^i_Σ converge to a limit. One can assume that σ is parametrized by arc length. Let Σ' be a subdivision which is finer than Σ, with $t'_l = t_k$ and $t'_m = t_{k+1}$. Let us estimate the contribution of $[t_k, t_{k+1}]$ to $A^i_{\Sigma'} - A^i_{\Sigma}$. Since $[x+y,z]_i - [x,z]_i$ is a homogeneous polynomial in the coordinates of x and y, linear with respect to z, there is a bound of the form

$$\|[x+y,z]_i - [x,z]_i\| \leqslant \text{const } \|z\| \sum_{q=1}^{i-1} \|y\|^q \|x\|^{i-1-q}$$

Setting $x = \sigma(t_k)$, $y = \sigma(t'_n) - \sigma(t_k)$, $z = \sigma(t'_{n+1}) - \sigma(t'_n)$, one obtains from the previous estimate

$$\begin{aligned} &\|[\sigma(t'_n), \sigma(t'_{n+1}) - \sigma(t'_n)]_i - [\sigma(t_k), \sigma(t'_{n+1}) - \sigma(t'_n)]_i\| \\ &\leqslant \text{const } \|t'_{n+1} - t'_n\| \sum_{q=1}^{i-1} \|\sigma(t_k)\|^{i-1-q} \|t'_{n+1} - t_k\|^q \end{aligned}$$

whence

$$\begin{aligned} &\left\| \sum_{n=l}^{m-1} [\sigma(t'_n), \sigma(t'_{n+1}) - \sigma(t'_n)]_i - [\sigma(t_k), \sigma(t'_{k+1}) - \sigma(t_k)]_i \right\| \\ &\leqslant \text{const } \|t_{k+1} - t_k\| \sum_{q=1}^{i-1} \|\Sigma\|^q t_k^{i-1-q} \end{aligned}$$

and then 15

$$\begin{aligned} \left\| A^i_{\Sigma'} - A^i_{\Sigma} \right\| &\leqslant \text{const. } \sum_{q>1} \|\Sigma\|^q \int_0^s t^{i-1-q} dt \\ &\leqslant \text{const. } \|\Sigma\| s \left(\|\Sigma\| + s \right)^{i-2}. \end{aligned}$$

4.6. **Definition.** The i-th *area swept* by the curve segment $[0, \sigma(t)]$ is the limit

$$\int_0^s [\sigma, d\sigma]_i = \lim_{\|\Sigma\| \to 0} A^i_{\Sigma},$$

where Σ goes over the subdivisions of $[0, s]$ with the mesh tending to 0. One has an estimate

$$\text{(o)} \qquad \left\| A^i_{\sigma} - \int_0^s [\sigma, d\sigma]_i \right\| \leqslant \text{const. } \|\Sigma\| s \left(\|\Sigma\| + s \right)^{i-2}.$$

4.7. **Proposition.** *Let σ be a curve parametrized by arclength in $\mathcal{G}$. For almost every s,*
$$\lim_{\varepsilon\to 0}\varepsilon^{-i}\int_s^{s+\varepsilon}[\sigma,d\sigma]_i=0.$$
Since it is Lipschitz with Lipschitz constant equal to 1, σ has an ordinary derivative almost everywhere, $\dot\sigma(s)$. By Egoroff and Lusin's theorems, for every $\varepsilon>0$, there is a closed subset $F\subset\mathbf{R}$ such that

(i) $\mathscr{H}^1(\mathbf{R}\setminus F)<\varepsilon$,
(ii) $a(\tau)=\sup\{\|(\sigma(s+r)-\sigma(s))/r-\dot\sigma(s)\| : s\in F\}$ goes to 0 when τ tends to 0.
(iii) the derivative $\dot\sigma$ is continuous on F.

Let us prove that the conclusion of Proposition 4.7 holds for every density point s of F. Assume for instance that 0 is such a point and denote
$$b(\varepsilon)=\sup\{\|\dot\sigma(t)-\dot\sigma(0)\| : t\in F, |t|<\varepsilon\}.$$
Let Σ be a subdivision of $[0,\varepsilon]$. Denote by K the set of indices k such that $[t_k,t_{k+1}]\cap F\neq\emptyset$.

- If $k\notin K$, then
$$\|\,[\sigma(t_k),\sigma(t_{k+1})-\sigma(t_k)]_i\,\|\leqslant t_k^{i-1}(t_{k+1}-t_k).$$
Setting
$$A^{-K}=\sum_{k\notin K}[\sigma(t_k),\sigma(t_{k+1})-\sigma(t_k)]_i\,,$$
one obtains
$$\|A^{-K}\|\leqslant\int_{[0,\varepsilon]\setminus F}t^{i-1}dt.$$
Let us denote by μ the measure of the set $[0,\varepsilon]\setminus F$. As the function $t\mapsto t^{i-1}$ is nondecreasing,
$$\int_{[0,\varepsilon]\setminus F}t^{i-1}dt\leqslant\int_{\varepsilon-\mu}^{\varepsilon}t^{i-1}dt=\frac{1}{i}\left(1-\left(1-\frac{\mu}{\varepsilon}\right)^i\right)\varepsilon^i.$$
Since 0 is a density point of F, the ratio μ/ε goes to 0 when ε goes to 0, hence $A^{-K}=o(\varepsilon^i)$.
- Otherwise, $k\in K$. Let us select $u_k\in[t_k,t_{k+1}]\cap F$. Then
$$\left\|\frac{\sigma(t_{k+1})-\sigma(u_k)}{t_{k+1}-u_k}-\dot\sigma(u_k)\right\|\leqslant a(\|\Sigma\|)$$
and
$$\left\|\frac{\sigma(u_k)-\sigma(t_k)}{u_k-t_k}-\dot\sigma(u_k)\right\|\leqslant a(\|\Sigma\|)$$

so that

$$\left\| \frac{\sigma(t_{k+1}) - \sigma(t_k)}{t_{k+1} - t_k} - \dot{\sigma}(u_k) \right\| \leqslant a(\|\Sigma\|).$$

Now set

$$\begin{aligned} A^K &= A^i_\Sigma - A^{-K} \text{ and} \\ B &= \sum_{k \in K} [\sigma(t_k), \dot{\sigma}(u_k)]_i (t_{k+1} - t_k). \end{aligned}$$

Then (summing the inequalities above)

$$\begin{aligned} \|A^K - B\| &\leqslant \text{const.} \sum_{k \in K} a(\|\Sigma\|) t_k^{i-1} (t_{k+1} - t_k) \\ &\leqslant \text{const.}\, a(\|\Sigma\|) \varepsilon^i. \end{aligned}$$

As $0 \in F$,

$$\begin{aligned} &\left\| \frac{\sigma(t_k)}{t_k} - \dot{\sigma}(0) \right\| \leqslant a(t_k) \leqslant a(\varepsilon) \text{ and} \\ &\|\dot{\sigma}(u_k) - \dot{\sigma}(0)\| \leqslant b(\varepsilon). \end{aligned}$$

Let us write

$$[\sigma(t_k), \dot{\sigma}(u_k)]_i = [\sigma(t_k), [t_k \dot{\sigma}(0) + \text{error term}, \dot{\sigma}(0) + \text{error term}]]_{i-1} \,.$$

Then 17

$$\begin{aligned} \|[\sigma(t_k), \dot{\sigma}(u_k)]_i\| \leqslant \text{const.}\, \|\sigma(t_k)\|^{i-2} \big(& t_k b(\varepsilon) \\ & + \|\sigma(t_k)\| a(\varepsilon) + t_k a(\varepsilon) b(\varepsilon) \big) \,, \end{aligned}$$

hence

$$\begin{aligned} \|B\| &\leqslant \text{const.}\, (a(\varepsilon) + b(\varepsilon) + a(\varepsilon) b(\varepsilon)) \sum_{k=0}^{N-1} t_k^{i-1} (t_{k+1} - t_k) \\ &= o(\varepsilon^i) \end{aligned}$$

and this bound does not depend on the subdivision. Thus we bounded A^i_Σ with a term $o(\varepsilon^i)$ that does only depend on the mesh of σ; plus a term that goes to 0 when the mesh goes to zero. We can conclude from this bound that

$$\int_0^\varepsilon [\sigma, d\sigma]_i = o(\varepsilon^i).$$

4.8. **Proof of Proposition 4.1.** Let c be a rectifiable curve in a Carnot group N, that develops into σ in the Lie algebra. If all the swept areas are such that

$$\int_0^{\varepsilon}[\sigma, d\sigma]_i = o(\varepsilon^i),$$

then c is differentiable at 0.

As announced, let us write

$$c(\varepsilon) = c(0)\exp(),$$

and approximate $\sigma^i(\varepsilon)$ with σ^i_N where, given a subdivision Σ of $[0,\varepsilon]$, σ^i_k is the V^i component of

$$\log\prod_{m=0}^{k-1}\exp\left(\sigma(t_{m+1}) - \sigma(t_m)\right).$$

We will prove by induction on the integer i that there exists a family of functions ϕ^i_a depending on a parameter a such that

(i) ϕ^i_a is nondecreasing;
(ii) the limit $\phi^i(s) = \lim_{a\to 0}\phi^i_a(s)$ exists for all s, and it is such that $\lim_{s\to 0}\phi^i(s) = 0$;
(iii) for every integer k,

$$\|\sigma^i_k\| \leqslant \phi^i_{\|\Sigma\|}(t_k).$$

Define

$$\psi^i(s) = \sup\left\{\left\|\int_0^u[\sigma, d\sigma]_i\right\| : 0 \leqslant u \leqslant s\right\}.$$

The function ψ^i is nondecreasing and, by assumption, $\lim_{s\to 0} s^{-i}\psi^i(s) = 0$. The relation

$$\sigma^2_{k+1} = \sigma^2_k + \frac{1}{2}\left[\sigma(t_k), \sigma(t_{k+1}) - \sigma(t_k)\right]$$

implies that $\sigma^2_k = A^2_k$; so (i), (ii) and (iii) above hold with $\phi^2_a(s) = \psi^2(s) + Cas$, where C is the constant in Equation (o) from §4.6.

In general, the relation between σ^i_{k+1} and σ^i_k can be put under the form

$$\sigma^i_{k+1} - \sigma^i_k = [\sigma(t_k), \sigma(t_{k+1}) - \sigma(t_k)]_i + P(\sigma^{j<i}_k, \sigma(t_k), \sigma(t_{k+1}) - \sigma(t_k)),$$

where P is a polynomial map that is linear with respect to $\sigma(t_{k+1} - \sigma(t_k)$ and depends on σ^j_k for $1 < j < i$. So one may write

$$\sigma^i_k = A^i_k + \sum_{m=0}^{k-1} P\left(\sigma^{j<i}_m, \sigma(t_m), \sigma(t_{m+1}) - \sigma(t_m)\right),$$

whence

$$\|\sigma_k^i\| \leqslant \|A_k^i\| + \sum_{m=0}^{k-1} \widetilde{P}\left(\|\sigma_m^{j<i}\|, \|\sigma(t_m)\|\right)(t_{m+1} - t_m),$$

where $\widetilde{P}$ denotes a polynomial with positive, integer coefficients which is homogeneous, that is,

$$\widetilde{P}(\lambda^j \phi^j, \lambda t) = \lambda^{i-1}\widetilde{P}(\phi^j, t) \tag{-}$$

for all $\lambda > 0$ and $\widetilde{P}(0, t) = 0$ for all t. Assume that the ϕ_a^j are constructed for all $j > i$ and that they satisfy (i), (ii) and (iii). Then

$$\begin{aligned}\|\sigma_k^i\| &\leqslant \psi^i(t_k) + C\|\Sigma\| t_k(\|\Sigma\| + t_k)^{i-2} + \sum_{m=0}^{k-1} \widetilde{P}(\phi_{\|\Sigma\|}^{j<i}(t_m), t_m)(t_{m+1} - t_m) \\ &\leqslant \phi_{\|\Sigma\|}^i(t_k),\end{aligned}$$

where

$$\phi_a^i(s) = \psi^i(s) + \int_0^s \widetilde{P}\left(\phi_a^{j<i}(t), t\right) dt + Cas(a+s)^{i-2}.$$

Clearly, ϕ_a^i is nondecreasing, so that (i) and (iii) hold. By (-),

$$t^{-i+1}\widetilde{(}\phi_a^{j<i}(t), t) = \widetilde{P}(t^{-j}\phi_a^j, 1).$$

By the induction hypothesis, this tends to $\widetilde{P}(0, 1) = 0$ when t goes to 0. We may conclude that $\phi^i(s) = o(s^i)$. This finishes the proof.

5. Proof of Theorem 3

A Lipschitz map between Carnot groups has all the hypotheses of Corollary 3.3. Indeed, its local dilation is bounded. Moreover, the 19 subspace V^1 generates the Lie algebra $\mathcal{N}$ and for every $v \in V^1$ and $x \in N$, the curve $c(s) = x \exp(sv)$ is rectifiable. The image of this curve by a Lipschitz map is rectifiable, hence it is differentiable almost everywhere by Proposition 4.1. Thus we proved the part of Theorem 2 that concerns Lipschitz maps.

Let Γ and Γ' be discrete, virtually nilpotent groups. A quasiisometry from Γ to Γ' induces a bilipschitz homeomorphism between the tangent cones at infinity $\mathrm{gr}(\Gamma \otimes \mathbf{R})$ and $\mathrm{gr}(\Gamma' \otimes \mathbf{R})$ (abstract nonsense). Its differential is also bilipschitz, so it is an isomorphism from $\mathrm{gr}(\Gamma \otimes \mathbf{R})$ to $\mathrm{gr}(\Gamma' \otimes \mathbf{R})$ at almost every point. This finishes the proof of Theorem 3.

6. Quasiconformal homeomorphisms and rectifiability

We will now prove that a quasiconformal homeomorphism between Carnot groups also has the hypotheses of Corollary 3.3.

Let us first recall a few definition. When $B = B(x, r)$ is a ball in a metric space, it is convenient to write kB for the ball $B(x, kr)$ with the same center.

6.1. **Definition.** Let X be a metric space. An annulus in X is $(a, \widetilde{a})$ where $a \subset \widetilde{a}$. It is a k-annulus if there exists B such that

$$B \subset a \subset \widetilde{a} \subset kB.$$

Let X and X' be metric spaces and let η be a homeomorphism of $[0, +\infty)$ onto itself. A map $f \colon X \to X'$ is called η-quasisymmetric if it sends any k-annulus that is small enough in X into a $\eta(k)$-annulus of X'. A homeomorphism f between X and X' is called η-quasiconformal if f and f^{-1} are η-quasisymmetric.

We will first prove that if two Carnot groups are locally quasiconformal, then they have the same Hausdorff dimension. The argument is an avatar of the "length-area methode". We will use several times the following elementary lemma.

6.2. **Lemma.** [Fed69, p.143] *Let X be a metric space. Let $\{B_i, i \in I\}$ be a cover of X with balls. One can extract a subfamily $\{B_j, j \in J\}$ of disjoint balls such that the balls with the same center $\{3B_j, j \in J\}$ cover X.*

Heuristic proof. Choose the largest ball first, then the second largest that do not meet the previous one, etc. □

6.3. **Lemma.** *Let U be an open subset in a metric space X, equipped with two measures μ and ν. Let Γ be a curve family in X, equipped with a measure $d\gamma$. Let $p > 1$. Let us assume that, for every ball B of X that is contained in U,*

$$\int_{\{\gamma\in\Gamma:\gamma\cap B\neq\emptyset\}} d\gamma \leqslant \mu\left(\frac{1}{3}B\right)^{1-1/p}.$$

For every ball $B \subset U$, we set

$$\phi(B) = \nu\left(\frac{1}{3}B\right)^{1/p}$$

and we denote Φ^1 the 1-dimensional measure obtained by Carathéodory's construction from ϕ (see below). Then

$$\int_\Gamma \Phi^1(\gamma)d\gamma \leqslant \nu(U)^{1/p}\mu(U)^{1-1/p}.$$

Let us recall that

$$\Phi^1(\gamma) = \limsup_{\varepsilon\to 0} \Phi^1_\varepsilon(\gamma)$$

where $\Phi^1_\varepsilon(\gamma)$ is the infimum of

$$\sum_i \phi(B_i)$$

over the covers by balls B_i with radius less or equal ε [Fed69, p.170].

Start from a covering of U by balls $\widetilde{B}_i = \frac{1}{3}B_i$ contained in U; with diameter less than ε, and let us apply Lemma 6.2 in order to obtain a cover $\{B_i\}$ such that the balls $\frac{1}{3}B_i$ do not overlap.

For every $\gamma \in \Gamma$, denote

$$1_i(\gamma) = \begin{cases} 1 & B_i \cap \gamma \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \Phi^1_\varepsilon(\gamma) &\leqslant \sum_i 1_i(\gamma)\phi(B_i) \\ &= \sum_i 1_i(\gamma)\nu(\widetilde{B}_i)^{1/p} \end{aligned}$$

whence, integrating and applying the assumption on μ and then Hölder's inequality, 21

$$\begin{aligned} \int_\Gamma \Phi^1_\varepsilon(\gamma)d\gamma &\leqslant \sum_i \left(\int_\Gamma 1_i(\gamma)d\gamma\right)\nu(\widetilde{B}_i)^{1/p} \\ &\leqslant \sum_i \nu(\widetilde{B}_i)^{1/p}\mu(\widetilde{B}_i)^{1-1/p} \\ &\leqslant \left(\sum_i \nu(\widetilde{B}_i)\right)^{1/p}\left(\sum_i \mu(\widetilde{B}_i)\right)^{1-1/p} \\ &\leqslant \nu(U)^{1/p}\mu(U)^{1-1/p} \end{aligned}$$

since the $\widetilde{B}_i$ are disjoint. As $\Phi^1_\varepsilon(\gamma)$ increases when ε decreases to 0, we can take a limit as $\varepsilon \to 0$.

6.4. **Notation.** The theory of measure differentiation can be applied to balls of Carnot-Carathéodory metrics, see [Ste70] or [Fed69, page 152]. Indeed, a sufficient condition for that is that for every ball B, $\mathscr{H}^p(2B) \leqslant \mathrm{const.}\,\mathscr{H}^p(B)$. Especially, if f is a homeomorphism between open subsets of Carnot groups with Hausdorff dimension p and p', one can define its Jacobian

$$f'(x) = \lim_{\varepsilon \to 0} \frac{\mathscr{H}^{p'}(f(B(x,\varepsilon)))}{\mathscr{H}^p(B(x,\varepsilon))}$$

which exists and is finite almost everywhere.

6.5. **Proposition.** *Let N and N' be Carnot groups. Let f be a quasiconformal homeomophism between open subsets of N and N' respectively. Then*

- *N and N' have equal Hausdorff dimension,*
- *f sends almost every orbit of a left-invariant horizontal vector field on a rectifiable curve*
- *The local dilation Lip_f is almost everywhere finite:*

$$\left(\mathrm{Lip}_f\right)^p \leqslant \eta(1)^p f'.$$

Let U be an open subset of a Carnot group of Hausdorff dimension p, $\mu = \mathscr{H}^p$, $v \in V^1$ a horizontal, left-invariant vector field. Consider the curve family Γ of orbits of v.

Let ω be a biinvariant volume form. Then $\iota_v\omega$ is closed hence it is basic, and defines a measure $d\gamma$ on the space of orbits that is invariant through left translation, homogeneous with degree $p-1$ under the dilations $e^{t\alpha}$. Hence, up to a normalization of ω, one has, for every ball B contained in U,

$$\int_{\{\gamma\in\Gamma:\gamma\cap B\neq\emptyset\}} d\gamma = \mu\left(\frac{1}{3}B\right)^{1-1/p},$$

and the assumptions of Lemma 6.3 hold.

Let $f\colon U \to U' \subset N'$ be a quasiconformal homeomorphism between open subsets of finite volume. Denote $\nu = (f^{-1})_*\mathscr{H}^{p'}$. Let $B \subset U$ be a ball. If f is η-quasisymmetric, there exists a ball B' in N' such that

$$B' \subset f\left(\frac{1}{3}B\right) \subset f(B) \subset \eta(3)B'.$$

Then

$$
\begin{aligned}
\phi(B) &= \nu\left(\frac{1}{3}B\right) \\
&= \left(\mathscr{H}^{p'} f(\frac{1}{3}B)\right)^{1/p} \\
&\geqslant \mathscr{H}^{p'}(B')^{1/p} \\
&\geqslant \sigma^{p'/p}\,\mathrm{radius}(B')^{p'/p} \quad \text{for some } \sigma > 0 \\
&\geqslant \left(\frac{\sigma}{\eta(3)}\right)^{p'/p} \mathrm{radius}(\eta(3)B')^{p'/p}
\end{aligned}
$$

since there exists a constant σ such that, for all $x' \in U'$ and for all $r > 0$,

$$\mathscr{H}^{p'}(B'(x', r)) \geqslant (\sigma r)^{p'}.$$

Assume that the B_i cover γ. Then the $\eta(3)B_i'$ cover $f(\gamma)$ and we conclude that

$$\Phi^1(\gamma) \geqslant \left(\frac{\sigma}{\eta(3)}\right)^{p'/p} \mathscr{H}^{p'/p} f(\gamma).$$

By Lemma 6.3, the image by f of amlost every orbit of v has a finite (p'/p)-dimensional Hausdorff measure. Especially, $p' \geqslant p$.

Let us set, for al small $r > 0$,

$$
\begin{aligned}
L &= \sup\{d(f(x), f(y) : d(x, y) < r\} \\
l &= d(f(x), f(\partial B(x, r)).
\end{aligned}
$$

when r goes to 0, $\mathrm{Lip}_f(x) = \limsup L/r$ and $L/l \leqslant \eta(1)$. Thus, for r small enough, 23

$$
\begin{aligned}
\left(\frac{L}{r}\right)^p &\leqslant \eta(1)^p \left(\frac{l}{r}\right)^p \\
&\leqslant \eta(1)^p \frac{\mathscr{H}^p(f(B(x, r))}{r^p}.
\end{aligned}
$$

At almost every point x, the right hand side goes to $\eta(1)^p f'(x)$, qed.

This finishes the proof of Theorem 2.

7. Quasiconformal homeomorphisms are absolutely continuous

We prove here that quasiconformal homeomorphsisms between Carnot groups of dimension > 1 send zero-measure sets on zero-measure sets. We will deduce from this fact that the differential is almost everywhere a group isomorphism. We use Gehring's method [Geh62, p. 377] that we transpose in the spirit of Lemma 6.3.

7.1. **Lemma.** *We keep notation as in Lemma 6.3, and require in addition that there exists a constant ρ such that, for every ball B, $\mu(2B) \leqslant \rho\mu(B)$. Let us define*

$$E = \left\{\frac{d\nu}{d\mu} \neq 0\right\}.$$

Then

$$\int_\Gamma \Phi^1(\gamma)d\gamma \leqslant \nu(E)^{1/p}\mu(E)^{1-1/p}.$$

We can assume that $\mu(U) < +\infty$. The assumption on μ ensures that, at almost every $x \in U$,

$$\frac{d\nu}{d\mu}(x) = \lim_{r\to 0} \frac{\nu B(x,r)}{\mu B(x,r)}$$

(see [Fed69, 2.9]). Let K be a compact subset of $U \setminus E$, and let $\varepsilon > 0$. For every $x \in K$, let us choose a ball B_x, centered at x, with a radius at most ε, such that $\nu(1/3B_x) \leqslant \varepsilon^p\mu(1/3B_x)$.

If $x \notin K$, let us pick B_x, with radius less than ε, not intersecting K. By 6.2 we can extract from this family of balls a cover by balls B_i such that the $\widetilde{B}_i = \frac{1}{3}B_i$ are pairwise disjoint. Let us denote

$$J = \{i : B_i \text{ is centered on } K\}.$$

Then

$$\begin{aligned}
\int_\Gamma \Phi^1_\varepsilon(\gamma)d\gamma &\leqslant \sum_{i\in J} \phi(B_i)\mu(\widetilde{B}_i)^{1-1/p} + \sum_{i\in I\setminus J} \phi(B_i)\mu(\widetilde{B}_i)^{1-1/p} \\
&\overset{\varepsilon}{\leqslant} \sum_{i\in J} \mu(\widetilde{B}_i) + \sum_{i\in I\setminus J} \phi(B_i)\mu(\widetilde{B}_i)^{1-1/p} \\
&\leqslant \varepsilon\mu(U) + \nu(U\setminus K)^{1/p}\mu(U\setminus K)^{1-1/p}.
\end{aligned}$$

Letting ε go to 0 and K tend to E, we obtain the inequality.

7.2. **Corollary.** *Let X and X' be metric spaces equipped with measures μ and ν. Let $p > 1$ be a real number. Assume that there exist constants ρ, σ, τ, v such that, if $B \subset X$ and $B'(x', r) \subset X'$ are two balls, then*

$$\mu(2B) \leqslant \rho\mu(B)$$

and

$$(\sigma r)^p \leqslant \nu B'(x', r') \leqslant (\tau r)^p.$$

Let Γ be a family of curves in X, equipped with a measure $d\gamma$. Assume that for every ball B of X,

$$v\mu(B)^{1-1/p} \leqslant \int_{\{\gamma\in\Gamma:\gamma\cap B\neq\emptyset\}} d\gamma \leqslant \mu(B)^{1-1/p}.$$

Then for every quasiconformal homeomorphism $f\colon X \to X'$, the measure $f_*\mu$ is absolutely continuous with respect to ν.

Since f is quasisymmetric, for every $\gamma \in \Gamma$, one has (with the notation as in Lemma 6.5)

$$\Phi^1(\gamma) \geqslant \frac{\sigma}{\eta(3)}\mathscr{H}^1 f(\gamma).$$

Let $x \in X$ and let $r > 0$. As f^{-1} is quasisymmetric, there exists a ball B centered at x such that

$$\eta(2)^{-1}B \subset f^{-1}B'(f(x),r) \subset f^{-1}B'(f(x),2r) \subset B.$$

If γ intersects with $\eta(2)^{-1}B$ but is not contained in B, $f(\gamma)$ joins $B'(f(x),r)$ to the complementary subset of $B'(f(x),2r)$, hence its length is at least r, and

$$\Phi^1(\gamma) \geqslant \frac{\sigma}{\eta(3)}r.$$

Let us apply Lemma 7.1 to the open set B, equipped with the measures μ and $(f^{-1})_*\nu$, and the curve family

$$\Gamma(B) = \left\{\gamma \in \Gamma : \gamma \cap \eta(2)^{-1}B \neq \emptyset\right\}.$$

Thus

$$r\frac{\sigma}{\eta(3)}\int_{\Gamma(B)} d\gamma \leqslant \nu(f(B))^{1/p}\mu(E\cap B)^{1-1/p}.$$

Since f is quasisymmetric, there exists a ball $B' \subset X'$ such that

$$B' \subset f\left(\eta(2)^{-1}B\right) \subset f(B) \subset \eta\circ\eta(2)B',$$

whence

$$\nu(f(B))^{1/p} \leqslant \nu(\eta\circ\eta(2)B')^{1/p} \leqslant \tau\eta\circ\eta(2)r$$

because $B' \subset B(f(x),r)$.

Besides, note that

$$\int_{\Gamma(B)} d\gamma \geqslant v\mu(B)^{1-1/p}.$$

Combining these inequalities one gets

$$\frac{\mu(E\cap B)}{\mu(B)} \geqslant \text{const.}^{1-1/p}$$

where the constant only depends on constant in the statement and the function η.

If $p > 1$, this inequality prevents x from being a density point of $X \setminus E$. We can deduce from this that the Jacobian f' of f is almost everywhere nonzero. For every $A \subset X$, we have, in general, the inequality

$$\nu(f(A)) \geqslant \int_A f' d\mu;$$

in particular, if $\nu(f(A)) = 0$ then f' vanishes almost everywhere on A, so that $\mu(A) = 0$: otherwise said, we proved that f is absolutely continuous.

7.3. **Proposition.** *Let f be a quasiconformal homeomorphism between open subsets of Carnot group with dimension greater than one. Then, f is absolutely continuous, and its differential is a group isomorphism almost everywhere.*

The assumptions of Corollary 7.2 are satisfied, as we saw in §6.5. At almost any point x, the differential $Df(x)$ exists and the Jacobian $f'(x)$ is non-zero. Let x be such a point. Set

$$f_t(\mu) = e^{t\alpha'} \left(f(x)^{-1} f(xe^{-t\alpha}\mu) \right).$$

By assumption, the maps f_t converge uniformly to $Df(x)$. Consequently, if B denotes the unit ball in the group,

$$Df(x)(B) \supset \cap_{T \to +\infty} \cup_{t > T} f_t(B).$$

If $Df(x)$ is not surjective, then

$$\begin{aligned} 0 &= \mathscr{H}^p Df(x)(B) \\ &\geqslant \lim_{t \to +\infty} \mathscr{H}^p f_t(B) \\ &= \lim_{t \to +\infty} e^{tp} \mathscr{H}^p f(B(e^{-t})) = f'(x). \end{aligned}$$

The next step in the study of the regularity of quasiconformal transformations is the so-called ACL property, absolute continuity along almost any "line". In [Mos73] this property is established before the absolute continuity, however the proof relies on the existence, in the sphere at infinity of a symmetric space of rank one, of a very particular family of curves (I thank U. Hamenstädt for pointing it out to me). B. Fuglede [Fug57] has shown more generally that a quasiconformal homeomorphism of Euclidean space is absolutely continuous on "almost any curve", i.e., except on a family of curves of zero modulus. We are going to prove directly a related result, where the notion of coarse modulus introduced in [Pan89] intervenes.

7.4. **Definition.** Let X be a metric space. A subset $a \subset X$ is a k-ball if (a, a) is a k-annulus (as in 6.1), that is, if there exists a ball B such that $B \subset a \subset kB$.

Let ϕ be a function taking positive values on the power set of X. For $l \geqslant 1$, define a new function on the power set of X by setting

$$\widetilde{\phi}_l(a) = \sup\{\phi(\widetilde{a})\colon (a, \widetilde{a}) \text{ is a } l\text{-annulus}\}$$

Let $p \geqslant 0$. We denote by $\Phi^{p,k}$ the measure obtained by Caratheodory's construction, summing ϕ (resp. $\widetilde{\phi}_l$) on the k-balls.

Let Γ be a curve family in X, and let $p \geqslant 0$, $k \geqslant 1$, $l \geqslant k$ be real numbers. We define the *coarse modulus* of Γ as the collection of numbers

$$M^{p,k,l,m}(\Gamma) = \inf \widetilde{\phi}_l^{p,k}$$

where the infimum is taken over all functions ϕ as above such that for every $\gamma \in \Gamma$, $\Phi^{1,m}(\gamma) \geqslant 1$.

7.5. **Propositions.** The following facts are established in [Pan89].

- The notion of "almost every curve" is preserved by quasiconformal homeomorphisms.[14]
- In a Carnot group, let Γ be a family of orbits of a horizontal vector field $v \in V^1$. If $\int_\Gamma d\gamma > 0$, then

  $$M^{p,k,l,m}(\Gamma) > 0$$

  for every $k \geqslant 1$, $l > 4k$, $m \geqslant 1$. Especially, if a property is satisfied by almost every curve, then it is satisfied by almost every orbit of v.

7.6. **Lemma.** *Let X and X' be metric spaces equipped with measures μ and ν. Let us assume that there exists constants σ and τ such that, for every ball $B' \subset X'$,*

$$\mu(B) \leqslant (\tau \operatorname{diameter}(B))^{1/p},$$

and

$$(\sigma \operatorname{diameter}(B'))^{1/p} \leqslant \nu(B').$$

Let $f\colon X \to X'$ be a η-quasiconformal homeomorphism such that $f^*\nu$ is absolutely continuous with respect to μ. Then, the restriction of f to almost every curve is absolutely continuous with respect to 1-dimensional Hausdorff measures.

Without loss of generality, one can assume $\nu(X') < +\infty$. Let γ be a rectifiable curve in X. Let us denote by ρ the positive measure given by $\rho(E) = \mathscr{H}^1 f(E)$ on γ. Its Radon-Nikodym decomposition can be written

$$\rho = u\mathscr{H}^1_{|\gamma|} + \rho_s$$

where ρ_s is singular with respect to $\mathscr{H}^1$. Let Γ_n be the family of curves γ such that $\rho_s(\gamma) \geqslant 1/n$. The family of curves along which γ is not absolutely continuous is the union of the families Γ_n, and we will prove that, for every $l, m \geqslant 1$, $M^{p,1,l,m}(\Gamma_n) = 0$.

Let R be a positive real number. Denote $\chi(t) = \max\{t/R\}$. Consider the function ϕ defined on the subsets a of U as

$$\phi(a) = \chi\left(\frac{\text{diameter}\, f(a)}{\text{diameter}(a)}\right) \text{diameter}\, f(a).$$

First note that as $t \leqslant R + t\chi(t)$ for all $t > 0$,

$$\text{diam}\, f(a) \leqslant R\, \text{diam}(a) + \phi(a).$$

On the other hand, for all $0 < s < t$, $\chi(t)/\chi(s) \leqslant t/s$, we have

$$\widetilde{\phi}_l(a) \leqslant \eta(l)^2 \phi(a). \tag{1}$$

Let us prove that for every curve γ in X and for all interval E of γ,

$$\mathscr{H}^1 f(E) \leqslant \text{const.}(R\mathscr{H}^1(E) + \Phi^{1,m}(E)).$$

Let a_i be m-balls covering E, with diameter $\leqslant \varepsilon$, such that

$$\sum_i \phi(a_i) \leqslant \Phi^{1,m}(E) + \varepsilon.$$

By assumption, there are balls B_i such that

$$\frac{1}{m} B_i \subset a_i \subset B_i.$$

Let us keep only the balls that intersect E. As in Lemma 6.2, let us extract from the covering $\{3B_i\}$ a subcovering with disjoint balls, such that the $9B_i$ cover E. By (1),

$$\phi(9B_i) \leqslant \eta(6m)^2 \phi(a_i).$$

When ε is small enough, E is not contained in any of the $3B_i$. However, it does intersect B_i. Thus

$$\text{diameter}(B_i) \leqslant \mathscr{H}^1(E \cap 3B_i)$$

whence

$$\sum_i \text{diameter}(B_i) \leqslant \mathscr{H}^1(E)$$

and so

$$\begin{aligned}
\mathscr{H}^1_{\varepsilon'}(f(E)) &\leqslant \sum_i \operatorname{diameter} f(9B_i)\\
&\leqslant \sum_i R \operatorname{diameter}(9B_i) + \phi(9B_i),\\
&\leqslant 9R\mathscr{H}^1(E) + \eta(6m)^2\Phi^{1,m}(E) + \varepsilon.
\end{aligned}$$

We just proved that for every rectifiable curve γ,

$$\rho_s(\gamma) \leqslant \eta(6m)^2\Phi^{1,m}(\gamma).$$

Consequently, for evey p, k, l, m, n,

$$M^{p,k,l,m}(\Gamma_n) \leqslant n^p\eta(6m)^{2p}\widetilde{\phi}_l^{p,1}(X).$$

In view of (1), it is sufficient to prove that $\phi^{p,1}(X)$ tends to 0 when R tends to $+\infty$. 29

For every $x \in X$, let us denote

$$\Theta(x) = \limsup_{B\to x} \frac{\nu f(B)}{\mu(B)}.$$

Let $A(t) = \{x \in X \mid \Theta(x) \geqslant t^p\}$ and $\omega(t) = \nu f(A(t))$. Since f is absolutely continuous, $\omega(t)$ tends to 0 when t tends to $+\infty$.

Fix $\varepsilon > 0$. For every $x \notin A(t)$, there exists a ball B_x centered at x with a radius less than ε such that

$$\nu f(3B_x) \leqslant t^p\mu(3B_x);$$

we then have

$$\phi(3B_x) \leqslant C\frac{t^2}{R}\nu f(3B_x)^{1/p},$$

where $C = C(\eta, \sigma, \tau)$ only depends on the constants in the statement.

Indeed, there is a ball B' in X' such that

$$B' \subset f(3B_x) \subset \eta(1)B'.$$

and then

$$\begin{aligned}
\frac{\sigma}{\eta(1)} \operatorname{diameter} f(3B_x) &\leqslant \sigma \operatorname{diameter}(B')\\
&\leqslant \nu(B')^{1/p}\\
&\leqslant t\mu(3B_x)^{1/p}\\
&\leqslant t\tau \operatorname{diameter}(3B_x)
\end{aligned}$$

and, as $\chi(t) \leqslant t/R$,

$$\phi(3B_x) \leqslant \frac{3\tau\eta(1)^2 t^2}{\sigma^2 R}\nu f(B_x)^{1/p}.$$

Again by Lemma 6.2 we extract a subfamily $\{3B_i\}$ which covers $X \setminus A(t)$ such that the B_i are disjoint, and we conclude that

$$\begin{aligned}\Phi_{\varepsilon}^{p,1}(X \setminus A(t)) &\leqslant \sum_i \phi(3B_i)^p \\ &\leqslant \left(\frac{Ct^2}{R}\right)^p \sum_i \nu f(B_i) \\ &\leqslant \left(\frac{Ct^2}{R}\right)^p \nu f(X).\end{aligned}$$

As $\chi(t) \leqslant 1$, we always have $\sigma\phi(B) \leqslant \eta(1)\nu f(B)^{1/p}$, and then

$$\Phi^{p,1}(A(t)) \leqslant \frac{\eta(1)}{\sigma}\nu f(A(t));$$

it follows that

$$\Phi^{p,1}(U) \leqslant \inf_{t>0} \left(\frac{Ct^2}{R}\right)^p \nu f(X) + \frac{\eta(1)}{\sigma}\omega(t)$$

which tends to 0 as R tends to $+\infty$.

This finishes the proof.

Lemma 7.6 applies to the open subsets of Carnot groups, and we can conclude the following.

7.7. **Proposition.** *A quasiconformal homeomorphism between open subsets of Carnot groups with Hausdorff dimension $p > 1$ is absolutely continuous on almost every line.*

The absolute continuity on the lines allows to replace the coarse modulus, that is a rather qualitative invariant, with finer invariants, the ordinary modulus or the capacity. These allow to globally control a transformation from “almost everywhere” information.

7.8. **Definition.** Let U be an open subset in a Carnot group. $ACL^p(U)$ is the space if continuous functions u on U which are absolutely continuous on almost every line, and whose local dilation Lip_u is in $L^p(U)$.

A *capacitor* is an open subset C together with two subsets $\partial_0 C$ and $\partial_1 C$ of $\overline{C}$, called *plates*.

The p-capacity of $(C, \partial_0 C, \partial_1 C)$ is

$$\inf \int_C \mathrm{Lip}_u^p \, d\mathscr{H}^p$$

the infimum being taken over functions $u \in ACL^p(C) \cap C^0(\overline{C})$ such that $u = 0$ on ∂C_0 and $u = 1$ on $\partial_1 C$. 31

7.9. **Example.** Let N be a Carnot group. Equip V^1 with a Euclidean norm. Let $v \in V^1$ be a unit vector, consider its orthogonal hyperplane $v^\perp \subset V^1$, and define $W = v^\perp \oplus [\mathcal{N}, \mathcal{N}]$. Let $F \subset W$ and $x \in X$.

The capacity of the "cubic" capacitor

$$C = \{\gamma_w(s) \colon w \in F, 0 \leqslant s \leqslant r\}, \textit{ where } \quad \gamma_w(s) = x \exp(w) \exp(sv),$$
$$\partial_0 C = \{\gamma_w(0) \colon w \in F\} \textit{ and } \qquad \partial_1 C = \{\gamma_w(r) \colon w \in F\}$$

is equal to $r^{-p} \mathscr{H}^p(C)$.

The proof is again obtained by an application of the length-area method.

Let $u \in ACL^p(C)$ be a function that is identically 0 and 1 on the plates. By Lemma 7.5 we have, for almost every $w \in F$,

$$\begin{aligned} 1 &= \int_0^r \frac{\partial}{\partial s} u \circ \gamma_w ds \\ &\leqslant \int_{\gamma_w} \mathrm{Lip}_u \, d\mathscr{H}^1 \\ &\leqslant r^{1-1/p} \left(\int_{\gamma_w} \mathrm{Lip}_u^p \, d\mathscr{H}^1 \right)^{1/p} \end{aligned}$$

whence

$$\begin{aligned} \int_C \mathrm{Lip}_u^p \, d\mathscr{H}^p &= c^{-1} \int_F \int_{\gamma_w} \mathrm{Lip}_u^p \, d\mathscr{H}^1 dw \\ &\geqslant c^{-1} r^{1-p} \int_F dw, \end{aligned}$$

(where c is the Jacobian $\frac{d\mathscr{H}^p}{dw} ds$) while

$$\mathscr{H}^p(C) = c^{-1} r \int_F dw.$$

Equality occurs with the function $u(y) = d(\pi(y), v^\perp)/r$ where

$$\pi \colon N \to N/[N, N] = V^1.$$

We can obtain other capacitors with positive capacity as follows. If C spearates the plates of C', then capacity$(C') \geqslant$ capacity(C). For instance, the spherical capacitor $C = B(x, R) \setminus B(x, r)$, $\partial_0 C = \partial B(x, r)$, $\partial_1 C = \partial B(x, R)$ has a nonzero capacity. By invariance of capcacity under dilations, one can see that this capacity $\tau(R/r)$ only depends on the ratio R/r. We do not need to know the exact values of τ. Note that there are other capacitors whose exact capacity is known; see [KR87].

7.10. **Definition.** [15] [16] A homeomorphism between open sets of Carnot groups is called 1-quasiconformal if it is η-quasiconformal for some function η and if its differential is a *similarity* almost everywhere, that is, its differential is the product of a dilation $e^{t\alpha}$ and a linear isometry from N to N'.

7.11. **Lemma.** *A 1-quasiconformal homeomorphism $f\colon U \to V$ between open subset of a Carnot group with Hausdorff dimension p induces an isometry from $ACL^p(U)$ to $ACL^p(V)$. Especially, it preserves capacities.*

Indeed, by Lemma 7.7 and the invariance of moduli by quasiconformal homeomorphisms (Lemma 7.5), the space ACL^p is preserved by η-quasiconformal homeomorphisms.

If $Df(x)$ is a similarity, then $f'(x) = \mathrm{Lip}_f(x)^p$, so that for every function[17] v on V,

$$\mathrm{Lip}_{v\circ f}(x)^p \leqslant \mathrm{Lip}_v(f(x))f'(x)$$

so that

$$\| \mathrm{Lip}_{v\circ f} \|_p \leqslant \| \mathrm{Lip}_v \|_p.$$

Part B. Quasiisometries of the quaternionic and Cayley hyperbolic spaces

8. Scheme of the proof of Theorem 1

In order to prove the Theorem 1 we are led to prove a result concerning the quasiconformal homeomorphisms of certain Carnot groups. Let X be a rank one symmetric space of noncompact type, let ∂X be its sphere at infinity. In §9.2 we endow ∂X of a conformal class of Carnot-Carathéodory metrics. More precisely, given a point $\infty \in \partial X$ we describe an analogue of the stereographic projection; this is a homeomorphism of $\partial X \setminus \{\infty\}$ on a Carnot group N endowed with a Carnot-Carathéodory metric. Extensions of isometries of X are conformal relative to this metric (§9.6) and the quasiisometries of X extend into quasiconformal homeomorphisms of ∂X (§9.12).

Thus, we need to prove that the only quasiconformal homeomorphisms of ∂X, when X is a quaternionic or Cayley hyperbolic plane, are the conformal transformation, i.e., those that come from isometries of X. By Part A, such a homeomorphism is almost everywhere differentiable, and the differential is a automorphism of N that commutes with the automorphisms $e^{t\alpha}$. We check in §10.1 that the centraliser of α in $\mathrm{Aut}(N)$

is the product of $\{e^{t\alpha}\}_{t\in\mathbf{R}}$ and a compact group. Thus a differentiable homeomorphism of ∂X is automatically 1-quasiconformal. Going from 1-quasiconformal to conformal should be achieved by applying a regularisation theorem. Since we lack such a theorem, we reproduce instead a global argument due to G.D. Mostow [Mos68, Lemma 12.2].

9. The sphere at infinity

9.1. For the algebraic description of the symmetric spaces with negative sectional curvature, we refer to [Mos73, Chapter 19]. Let us recall their classification. There are the hyperbolic spaces with constant curvature -1, that we denote $\mathbf{R}\mathrm{H}^n$, $n \geqslant 2$, the complex and quaternionic variants $\mathbf{C}\mathrm{H}^n$ and $\mathbf{H}\mathrm{H}^n$, $n \geqslant 2$, and a hyperbolic Cayley plane, $\mathbf{Ca}\mathrm{H}^2$. The latter have their curvature comprised between -4 and -1.

9.2. One can attach to these spaces, and to any simply connected Riemannan manifold with nonpositive sectional curvature as well, a sphere at infinity (See [EO73]). A *point at infinity* is an equivalence class of geodesic rays, where two geodesic rays $c, c' : [0, +\infty) \to X$ are equivalent if $d(c(t), c'(t))$ is bounded as $t \to +\infty$. One can show that, fixing one point x in X, the exponential map which associates to a unit tangent vector $u^i n T_x X$ the geodesic ray starting from x with initial speed u, is a homeomorphism between $T_x X$ and the set of points at infinity, or "sphere at infinity" ∂X. To any point at infinity one can associate a family of horospheres. This are orthogonal trajectories of the family of equivalent geodesic rays ending at this point. The *horofunctions* are the functions that are constant along horospheres, whose gradient is a unit vector.

9.3. It follows from the definition that the isometries act on the sphere at infinity. In the case of rank one symmetric spaces, the isometry group $G = \mathrm{Isom}(X)$ is transitive on ∂X, and the stabilizer of $\infty \in \partial X$ is a maximal parabolic subgroup. Let us denote by R the solvable radical of the maximal parabolic group G_∞ (in fact, R is a minimal parabolic). Then $N = [R, R]$ is nilpotent and acts simply transitively on $\partial X \setminus \infty$.

When the field $\mathbf{K}$ is $\mathbf{R}$, the group N is abelian. In the other case, the Lie algebra $\mathcal{N}$ admits a gradation

$$\mathcal{N} = V^1 \oplus V^2$$

where if we identify V^1 to $\mathbf{K}^{n-1}$ and V^2 to $\Im m\mathbf{K}$, the Lie algebra struc- 34 ture can be written down as

$$[(x_2, \dots, x_n), (y_2, \dots, y_n)] = \Im m \left(\sum_i \overline{x_i} y_i \right).$$

The plane fields V^1 and V^2 have a geometric interpretation: the orbits of N in X are horospheres attached t the point ∞. Brought onto a horosphere, the field V^2 is tangent to the $\mathbf{K}$-*lines*, and each vector of V^1 generates with the normal vector a totally real plane. Otherwise said, if ν is the vector that is normal to H, one has

$$V^1 = (\mathbf{K}\nu)^\perp, V^2 = TH \cap \mathbf{K}\nu.$$

9.4. The distributions V^1 and V^2 also reflect in the quantitative behavior of the Jacobi fields on X. Let c be a geodesic with endpoint at ∞. Then c is orthogonal to the N-orbits. Let us denote by H_t the N-orbit going through $c(t)$. For $v \in \mathcal{N}$, denote v_t the value at $c(t)$ of the corresponding Killing field. Along c, v_t is a Jacobi field.

If $v \in V_2$, then the v_t are tangent to the $\mathbf{K}$-line D containing c, and as D is totally geodesic with constant curvature -4, the norm of v_t is

$$\|v_t\| = e^{2t}\|v_0\|.$$

If $v \in V^1$, the v_t are orthogonal to D, the field v_t is tangent to a totally real plane, i.e. to a totally geodesic surface of constant curvature -1, hence

$$\|v_t\| = e^{t}\|v_0\|.$$

9.5. For every $t \in \mathbf{R}$, let us denote by A_t the translation of distance t along c. It is an isometry of X. Together with N, it generates the solvable group R. A_t normalises N and induces on it the automorphism $e^{t\alpha}$, where α is the derivation having matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

in the decomposition $\mathcal{N} = V^1 \oplus V^2$. Especially, N equipped with α is a Carnot group. The map

$$(\nu, t) \mapsto \nu \cdot c(t)$$

is a diffeomorphism from $N \times \mathbf{R}$ to X. In these coordinates, one has

$$A_s(n,t) = (e^{s\alpha}n, t+s),$$

and the metric of X (with sectional curvature between -4 and -1) can be written

$$g = g_t \oplus dt^2,$$

where the left-invariant metrics g_t on N have matrices of the form

$$\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix}$$

in the graduation $V^1 \oplus V^2$. When t tends to $+\infty$, the Riemannian metrics $e^{-2t} g_t$ on N do not converge to a Riemannian metric. They converge to a Carnot-Carathéodory metric, that we denote d_∞.

Since this metric is left-invariant on N, the metric transported to $\partial X \setminus \infty$ $n \mapsto n \cdot q$, $N \to \partial X \setminus \infty$

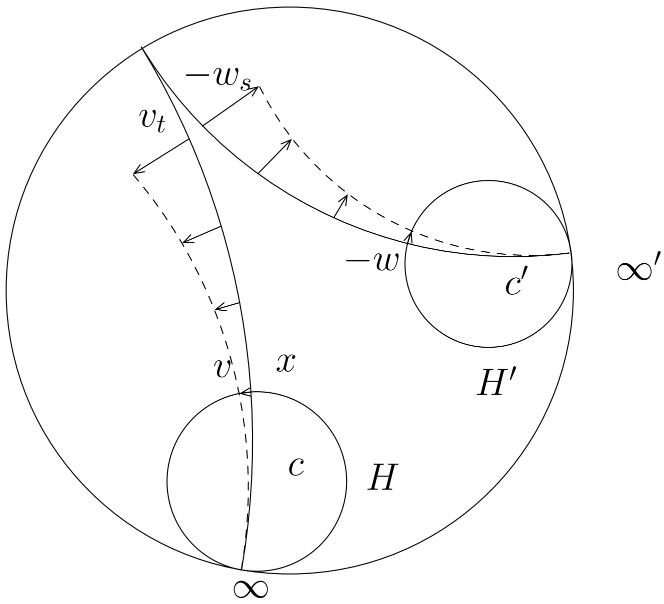

FIGURE 2.

does not depend on the choice of an origin q. Note that in the definition of d_∞, the choice of an origin on the geodesic c intervenes. An other choice leads to a proportional metric. This expresses the fact that the automorphism $e^{t\alpha}$ is a silimarity of ratio e^t for d_∞. More generally, one has the following.

9.6. **Lemma.** [Gro] *When the point at infinity ∞ is changed, the distances d_∞ are conformal, that is, the plane field V_1 does not depend on ∞ and the metric on V^1 stays conformal to itself.*

Fix two horospheres H and H' centered at ∞ and ∞'. Denote by $\pi\colon X \cup \partial X \setminus \infty \to H$ the orthogonal projection on H, and $\lambda : H \to \partial X \setminus \infty$ its inverse. We must prove that

$$\iota = \pi' \circ \lambda \colon H \to H'$$

preserves the plane field V^1 (orthogonal to the $\mathbf{K}$-lines) and is conformal on this plane field. We will prove that an approximation ι_t of ι almost 36 preserves V^1 and is almost conformal on this plane field. Let us denote $\lambda_t\colon H \to H_t$ the orthogonal projection on the horosphere at[18] a distance t of H, and set

$$\iota_t = \pi' \circ \lambda_t.$$

Let $x \in H$, and let c be the geodesic with origin x that is normal to H. Let $v \in V^1 \subset T_x H$. Let us denote v_t the Jacobi field such that $v_0 = v$ whose norm tends to 0 when t tends to $-\infty$. By[19] Lemme 6.2,

$$\|v_t\| = e^{-t}\|v_0\|.$$

By definition, $\lambda_{t*}v = v_t$. Set $w = \iota_{t*}v$, and let w_s be the Jacobi field along c'. Denote $s = d(c(t), H')$. Then w_s is the nearest point projection of v_t on the horosphere H'_s. As the curvature is negative and away from zero, when t is large, the angle ϕ between c and c' is small, thus

$$\left|\frac{w_s}{\|w_s\|} - \frac{v_t}{\|v_t\|}\right| \approx \phi,$$

hence, as v^1 and V^2 depend on the direction in a differentiable way,

$$d\left(\frac{w_s}{\|w_s\|}, V^1\right) \approx \phi.$$

Let us decompose, for every $u \in \mathbf{R}$, w_u as

$$w_u = w_u^1 + w_u^2 \in V^1 \oplus V^2.$$

Then,

$$\|w_0^2\|/\| = e^{-s}\|w_s^2\|/\|w_s^1\| \approx e^{-s}\phi,$$

which proves that ι_t preserves the plane field V^1. On the other hand,

$$\|w_0^1\| \approx e^{t-s}\|v_0\|$$

when t goes to $+\infty$, which proves that, on V^1, ι is conformal with ratio $e^{\theta-\theta'}$, where θ and θ' denote the horofunctions that vanish on H and H' (note that the difference of two horofunctions can be extended to the sphere at infinity).

9.7. **Extending quasiisometries to the boundary.** The notion of quasiisometry used here is due to Margulis [Mar75] ; however the idea of extending quasiisometries to the sphere at infinity probably goes back to Morse [Mor21] and certainly to Efremovitch [Efr52].

9.8. **Definition.** Let X and X' be metric spaces. A *quasiisometric embedding* of X into X' is a map $f\colon X \to X'$ such that

$$-C + \frac{1}{L} d(x,y) \leqslant d(f(x), f(y)) \leqslant L d(x,y) + C$$

where L and C are two constants. A *quasiisometry* of X onto X' is given by a pair of quasiisometric embeddings $f\colon X \to X'$ and $g\colon X' \to X$ such that, for all $X \in X$ and $x' \in X'$,

$$d(g \circ f(x), x) \leqslant C, d(f \circ g(x'), x') \leqslant C'.$$

A *quasigeodesic* in X is a quasiisometric embedding of $\mathbf{R}$ into X.

9.9. **Lemma.** *Let X be a Riemannian manifold with sectional curvature $K \leqslant -\mu^2 < 0$. For every quasigeodesic c of X, there exists a geodesic c' contained in a tubular neighborhood of c such that c is contained a tubular neighborhood of c'. The width $\tau(\mu, L, C)$ os these tubular neighborhoods only depends on L and C.*

For completeness we include a proof of this result that is otherwise folklore. By definition, for every $t, t' \in \mathbf{R}$,

$$-C + \frac{1}{L} \|t' - t\| \leqslant d(f(x), f(y)) \leqslant L\|t' - t\| + C.$$

This implies the following estimates where the length s of c between $x = c(t)$ and $y = c(t')$ intervenes:

$$(-) \qquad - C(1 + L^{-2}) + L^{-2} s$$

Fix two points x and y on c. Let z be the point in the part of c that is located between a and b which is the farthest from the geodesic segment σ between x and y. Fix a number $R > 0$.

Let a between z and x and b between z and y be the first points on c such that $d(a, \sigma) = d(b, \sigma) = R$. From a distance larger or equal to R, the projection onto σ contracts with a factor at least $\mathrm{ch}(\mu R)^{-1}$. Thus the projections a' and b' of a and b onto σ are such that

$$d(a', b') \leqslant \frac{s}{\mathrm{ch}(\mu R)},$$

where s is the length of the part of c between a and b. Especially,

$$d(a, b) \leqslant \frac{s}{\mathrm{ch}(\mu R)} + 2R.$$

By $(-)$,

$$d(a, b) \geqslant L^{-2} s C(1 + L^{-2});$$

in addition,

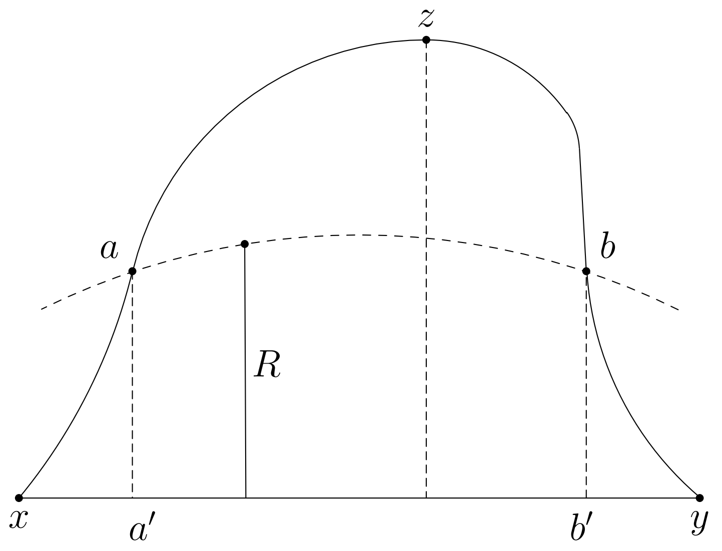

FIGURE 3.

$$\begin{aligned} s &\geqslant d(a,z)+d(z,b) \\ &\geqslant d(z,\sigma)-d(a,\sigma)+d(z,\sigma)-d(b,\sigma) \\ &\geqslant 2T-2R. \end{aligned}$$

Combining these inequalities, one finds that

$$T \leqslant R + \left(L^{-2} - \frac{1}{\mathrm{ch}(\mu R)}\right)^{-1}\left(R + \frac{C}{2}(1+L^{-2})\right).$$

We may now choose R such that, for instance, $\mathrm{ch}(\mu R) = 2L^2$. This gives an upper bound τ for T that only depends on μ, L and C. Finally, if $t_1 < t_2 < t_3$ we have

$$d(c(t_2), c([r_1, t_3])) \leqslant \tau(\mu, L, C),$$

so that the angle between the geodesic segments $[c(t_1), c(t_2)]$ and $[c(t_1), c(t_3)]$ is bounded above by a constant times $Te^{-\mu t/2}$. We may conclude that these geodesic segments converge to a geodesic c' which is such that

$$d(c, c') \leqslant \tau(\mu, L, C).$$

9.10. **Remark.** This Lemma shows that, in the definition of points at infinity, one could replace “geodesic” with “quasigeodesic”. As the quasi-isometric embeddings preserve quasigeodesics, thay extend to the sphere at infinity, and the quasiisometries extends into homeomorphisms between the spheres at infinity, a result originally due to V. Efremovitch and E. Tihomirova [ET64].

Similarly, we can define a topology on the sphere at infinity as follows: a neighborhood basis $U_{c,K}$ of a point at infinity ∞ is obtained by setting

$$U_{c,K} = \{\text{quasigeodesics } c' \mid d(c \cap K, c' \cap K) \leqslant 3\tau(\mu, L, C)\}$$

where c runs through all the L, C-quasigeodesics ending at ∞ and K runs through an exhaustion of X by compact subsets. It follows from the definition that if a sequence of uniformly quasiisometric embeddings (that is, a sequence of quasiisometric embeddings with the same constants L and C) $f_i \colon \partial X \to \partial X'$ converges uniformly[20], then the embeddings $f_i \colon \partial X \to \partial X'$ converge uniformly as well.

The next two Lemmas finish the reduction of Theorem 1 to a property of the sphere at infinity. The first Lemma shows that a quasiisometry is determined by its extension to the sphere at infinity up to the transformations that move points by a bounded amount.

9.11. **Lemma.** *Let $f, g \colon X \to X'$ be two (L, C)-quasiisometries between negatively curved manifolds with sectional curvature $K \leqslant -\mu^2$. If f and g have the same extension to the sphere at infinity ∂X, then for every $x \in X$,*

$$d(f(x), g(x)) \leqslant \tau'(\mu, L, C).$$

Let $\widetilde{g}$ be an inverse of g as in Definition 9.8. Considering the $(L^2, (L+1)C)$-quasiisometry $\widetilde{g} \circ f$ brings us back to the case when g is the identity map. Fix $x \in X$. Let c be the geodesic line through x that is orthogonal to the geodesic segment $[x, f(x)]$. By assumption, the quasigeodesic $f \circ c$ is asymptotic to c. So it lies in the $\tau(\mu, L, C)$-neighborhood of c. We thus have $d(x, f(x)) = d(f(x), c) \leqslant \tau(\mu, L, C)$. □

Finally, in the case of rank one symmetric spaces we defined in 9.6 a conformal class of metrics on ∂X. Thus we may qualify certain mappings between the boundaries as quasiconformal.

9.12. **Proposition.** *If f is a (L, C) quasiisometry between rank one symmetric spaces, its extension to the sphere at infinity is $H(L, C)$-quasiconformal.*

In fact, this is a special case of a general property of negatively curved manifolds with curvature bounded away from zero; see [Pan89]. In the case of symmetric spaces, it is particularly easy. Consider the family $\mathscr{F}$ of quasi-isometries of the form $h \circ f \circ g$ where g, h describe the isometries of X (resp. X'). By Ascoli, the family $\mathscr{F}$ is "proper" in the following sense: given $x \in X$ and a compact subset K of X', the subfamily of elements of $\mathscr{F}$ which send x into K is precompact for the topology of uniform convergence on the compacts of X.

It follows that the extension of $\mathscr{F}$ between the spheres at infinity has the so-called three-point property: if K_1, K_2 and K_3 are three disjoint closed subspaces of $\partial X'$ and x_1, x_2, x_3 are three distinct points of ∂X, then the subfamily of $\mathscr{F}$ which sends x_i into K_i for all i is precompact. Indeed the space of triples of distinct points in ∂X can be identified with the space of orthonormal pairs of tangent vectors of X which is the same as X as far as quasiisometries are concerned (Figure 4). This idea is due to J. Cheeger, see [Gro81a].

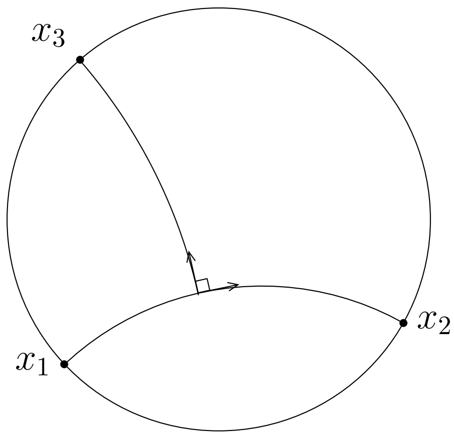

FIGURE 4.

We can identify $\partial X \setminus \infty$, resp. $\partial X' \setminus X'$ with a nilpotent group N, resp. N'. Let us check that the distorsion H_f is bounded at $x \in \partial X \setminus \infty$. As N acts on itself by conformal transformations, we may assume that x is the neutral element. Let us assume that $f(x) \neq \infty'$; here again, one may assume that $f(x)$ is the neutral element of N'. For every $\varepsilon > 0$, let g be the automorphism $e^{t\alpha}$ that sends the k-annulus $a_k(x, \varepsilon) = (B(x, \varepsilon), B(x, k\varepsilon))$ on the unit k-annulus $a_k(x, 1)$. Fix a point y in $\partial B(x, 1)$. Let h be the automorphism $e^{-s\alpha'}$ of N' such that

$$h \circ f \circ g(y) \in \partial B(f(x), 1).$$

When ε tends to 0, $g(y)$ tends to x, so h has a dilating effect in the neighborhood of $f(x)$. Especially $h(f(\infty))$ tends to ∞'. Setting $x_1 = x$, $x_2 = y$, $x_3 = \infty$, $K_1 = \{f(x)\}$, $K_2 = a_k(x, 1)$, $K_3 = \partial X' \setminus B(f(x), 2k)$, one sees that when ε tends to 0, the family of maps $h \circ f \circ g$ is precompact. Especially, the family[21] of subsets $h \circ f \circ g(\partial B(x, 1))$ is compact. It stays comprised between two balls $B(f(x), R)$ and $B(f(x), R)$, and does so uniformly with respect to x.

10. Graded automorphisms of maximal unipotent subgroups in rank one simple Lie groups

Let $X = \mathbf{K}\mathrm{H}^m$ be a noncompact symmetric space of rank one.

10.1. **Proposition.** [22] *If $\mathbf{K} = \mathbf{H}$ or $\mathbf{Ca}$, then any graded automorphism of $\mathcal{N}$ is a homothety, that is, the product of an isometry and an automorphism $e^{t\alpha}$.*

The Lie algebra $\mathcal{N}$ is nilpotent of class 2. It is graded by

$$\mathcal{N} = \mathbf{K}^{n+1} \oplus \Im m\mathbf{K}.$$

The subspace $\Im m\mathbf{K}$ is in the center, and if

$$x = (x_2, \ldots, x_m)$$
$$y = (y_2, \ldots, y_m) \in \mathbf{K}^{m-1},$$

then the Lie bracket is equal to

$$[x, y] = \Im m \left(\sum_{i=2}^{m} x_i \overline{y_i} \right);$$

see [Mos73, page 141]. We can endow the subspace $\mathbf{K}^{m-1}$ with the scalar product

$$\langle x, y\rangle = \Re e \left(\sum_{i=2}^{m} \overline{x_i} y_i \right).$$

A graded automorphism of $\mathcal{N}$ is given by two automorphisms $A \in \mathrm{Gl}_{\mathbf{R}}(\mathbf{K}^{m-1})$ and $B \in \mathrm{Gl}_{\mathbf{R}}(\Im m\mathbf{K})$ such that, if $x, y \in \mathbf{K}^{m-1}$,

$$[Ax, Ay] = B[x, y].$$

We must prove that A is the product of a homothety and an isometry of $\mathbf{K}^{m-1}$.

Let D denote the group of graded automorphisms of $\mathcal{N}$ with determinant equal to 1. We already now a large subgroup of D, namely the group of extensions of isometries of X that fix a geodesic. When $\mathbf{K} = \mathbf{H}$ the latter is $M = \mathrm{Sp}(m-1)\,\mathrm{Sp}(1)$ when $\mathbf{K} = \mathbf{H}$. When $\mathbf{K} = \mathbf{Ca}$ it is the subgroup M of $\mathrm{O}(\mathbf{Ca})$ that is the image of $\mathrm{Spin}(7)$ in the spin representation. We will prove that $D = M$. Let us first prove that the Lie algebras are the same, in the case of hyperbolic planes.

10.2. **Scholia.** *The Lie subalgebra $\mathfrak{so}(4)$ is maximal in $\mathfrak{sl}(4, \mathbf{R})$.*

Indeed, a subalgebra containing $\mathfrak{so}(4)$ is $\mathrm{SO}(4)$-invariant. But actually,[23]

$$\mathfrak{sl}(4) = \oplus_0^2 \mathbf{R}^4 = S_0^2 \oplus \Lambda^2 \mathbf{R}^4.$$

10.3. **Scholia.** *The Lie subalgebras of* $\mathfrak{sl}(8,\mathbf{R})$ *containing* $\mathfrak{spin}(7)$ *are* $\mathfrak{sl}(8,\mathbf{R})$, $\mathfrak{so}(8)$ *and* $\mathfrak{spin}(7)$.

Let us decompose $\mathfrak{sl}(8)$ into irreducible components under Spin(7). Denote V the spin representation of Spin(7). Then

$$\mathfrak{gl}(8) = V^* \otimes V.$$

The representation V of Spin(7) is associated to the fundamental weight ϖ_3. The representation $V^* \otimes V$, that is isomorphic to $V \otimes V$, contains an irreducible component U with dominant weight $2\varpi_3$. Weyl's character formula provides $\dim U = 35$ [Bou05]. The spin representation is a homomorphsim of $\mathfrak{so}(7)$ into $\mathfrak{sl}(8,\mathbf{R})$. Its image W is irreducible and has dimension 21. There is another subspace of $\mathfrak{gl}(8)$ that we know to be invariant under Spin(7): this is the subspace Z in $\mathfrak{sl}(8,\mathbf{R})$ formed by elements of $\Im m\mathbf{Ca}$ acting by Cayley multiplication (to the left or to the right) of dimension 7. It is not the trivial representation of Spin(7), and since Spin(7) has no irreducible representation of dimension < 7, the trivial one put aside, Z is isomorphic to the natural representation of SO(7). Especially, Z is irreducible. The decomposition of $\mathfrak{sl}(8,\mathbf{R})$ in irreducible components is thus

$$\mathfrak{sl}(8,\mathbf{R}) = U \oplus W \oplus Z$$

and the correspondig dimensions are

$$63 = 35 + 21 + 7.$$

This leaves only four possibilities for a subspace of $\mathfrak{sl}(8,\mathbf{R})$ containing W: it can be $W = \mathfrak{spin}(7)$, $W \oplus Z = \mathfrak{so}(8)$, $W \oplus Z \oplus U = \mathfrak{sl}(8,\mathbf{R})$, or $U \oplus W$, and we are left to proving that the last one is not a Lie subalgebra. The zero-trace symmetric matrices

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are such that

$$\frac{1}{2}[a,b] = i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us denote A, resp. B and I, the 8×8 matrix that one obtains by concatenating 4 diagonal blocs equal tp $\frac{a}{2}$, resp. to b and i. Then $A, B \in U$ and $[A,B] = I \in Z$, since I is the matric of the left multiplication by an element of $\Im m\mathbf{Ca}$ in the standard basis of the Cayley numbers. This finishes the proof that $U \oplus W$ is not a Lie subalgebra.

10.4. **Proof of Proposition 10.1 in the case $\mathbf{H}\mathrm{H}^2$ or $\mathbf{Ca}\mathrm{H}^2$.** Scholia 10.2 and Scholia 10.3 imply that M is the neutral component of D and thus D normalizes M. Since M has no outer automorphism, $D = Z_D(M)M$. But M is irreducible in $\mathbf{K}$, hence $Z_D(M) \subset \{+1, -1\} \subset M$, and then $M = D$.

10.5. **Proof of Proposition 10.1 in the case $\mathbf{H}\mathrm{H}^m$, $m \geqslant 2$.** Let A be a graded automorphism of $\mathcal{N}$, i;e., $A \in \mathrm{Gl}(\mathbf{H}^{m-1})$ and there exists $B \in \mathrm{Gl}(\Im m\mathbf{H})$ such that

$$[Ax, Ay] = B[x, y].$$

Note that for all $x \in \mathbf{H}^{m-1} \subset \mathcal{N}$ such that $x \neq 0$, the quaternionic line $x\mathbf{H}$ is exactly the bicommutant of x Indeed for every $x, y \in \mathbf{H}^{m-1}$,

$$[x, y] = 0 \iff y \perp x\Im m\mathbf{H},$$

so that the commutant $Z(x)$ contains the quaternionic hyperplane $P = (x\mathbf{H})^{\perp}$. As P is invariant when multiplied by imaginary numbers, any element y that commutes with P is orthogonal to P, hence it is in $x\mathbf{H}$.

It follows that A preserves the quaternionic lines. We may then use the fundamental theorem of affine geometry (see e.g. [Ber78]): A is $\mathbf{H}$-skew linear, that is, there exists a ring automorphism σ of $\mathbf{H}$ such that for every $x \in \mathbf{H}^{m-1}$ and $h \in \mathbf{H}$, $A(hx) = \sigma(x)A(x)$. Since the automorphisms of $\mathbf{H}$ are inner,

$$A \in \mathrm{Gl}(M-1, \mathbf{H})\,\mathrm{Sp}(1).$$

Up to composing A with an element of $\mathrm{Sp}(m-1)$, one may assume that A fixes a quaternionic line $\mathbf{H}x$. Note that the bracket $\Lambda^2\mathbf{H}x \to \Im m\mathbf{H}$ is surjective, consequently, $B \in \mathrm{End}(\Im m\mathbf{H})$ is determined by the restriction of A to the line $\mathbf{H}x$. By §10.4 in the case $X = \mathbf{H}\mathrm{H}^2$, one has $A \in CO(\mathbf{H}x)$, whence $B \in CO(\Im m\mathbf{H})$. Up to composing A on the right with a nonzero quaternion we may thus assume that A is $\mathbf{H}$-linear and B is the identity; i.e., A fixes every symplectic form $(x, y) \mapsto \langle x, yi\rangle$ for some $i \in \Im m\mathbf{H}$. Since A commutes with i, A is an isometry, thus $A \in \mathrm{Sp}(m-1)$. We proved that $A \in \mathrm{Sp}(m-1)\,\mathrm{Gl}(1, \mathbf{H})$ as announced. □

10.6. **Case $\mathbf{K} = \mathbf{R}$ and $\mathbf{C}$.** In the case of the real hyperbolic space, i.e., when the sectional curvature is constant, the group N is abelian, the derivation α is the identity, and the centraliser of α in N is the whole linear group. Besides, every diffeomorphism of the sphere is quasiconformal. In the case of the complex hyperbolic space, i.e., Kähler with constant holomorphic sectional curvature, N is a Heisenberg group, i.e.,

$\mathcal{N} = \mathbf{C}^{m-1} \oplus i\mathbf{R}$ and the bracket is given by the standard symplectic form on $\mathbf{C}^{m-1}$. The group of graded automorphisms is thus the conformal symplectic group $C\,\mathrm{Sp}(2m-2, \mathbf{R})$. Besides, every contact transformation is quasiconformal, see [KR85].

11. 1-quasiconformal transformations

11.1. **Definition.** [24] A homeomorphism between open subsets of Carnot groups is called 1-quasiconformal if it is quasiconformal and if its differential is a homothety almost everywhere.

It follows from part A that a quasiconformal homeomorphism of the sphere at infinity of a rank one symmetric space has a differential almost everywhere. In the case of quaternionic and Cayley hyperbolic spaces, this differential is a similarity by §10.

Hence we proved the following.

11.2. **Corollary.** *Let N be one of the Carnot groups at infinity of a quaternionic or Cayley hyperbolic space. Any quasiconformal homeomorphism between open subsets of N is* 1*-quasiconformal.*

Note that before one speaks of a 1-quasiconformal map on a Carnot group, one should specify a Carnot-Catathéodory metric. On the sphere at infinity of a rank one symmetric space, this was done in §9.6. □

In order to complete the proof of Theorem 1 we need to prove that such a transformation, when it is globally defined, is conformal, i.e., it comes from an isometry of the symmetric space. Let f be a 1-quasiconformal global homeomorphism of the spehre at infinity of a rank one symmetric space. We will prove that, after suitable normalization, f is an isometry fr a given Carnot-Carathéodory distance on a Carnot group. The method, that we borrow from G.D. Mostow [Mos68], consists in estimating the norm of the differential by means of the capacities, as they were defined in 7.8. Let us recall the following result from part A (this was Lemma 7.11).

11.3. **Lemma.** *Capacities are invariant through* 1*-quasiconformal homeormophisms.*

11.4. **Lemma.** *A* 1*-quasiconformal homeomorphism between open subsets of Carnot groups is locally Lipschitz.*

Let $f\colon U \to V$ be a 1-quasiconformal homeomorphism. Let $x \in U$. Fix $R < d(f(x), \partial V)$. Set $D = D(x) = d(x, f^{-1}\partial B(f(x), R))$. For $\varepsilon < D$, let us set $r = \max\{d(f(x), f(z) : d(x,z) \leqslant \varepsilon\}$, so that $\mathrm{Lip}_f(x) = \lim_{\varepsilon \to 0} r/\varepsilon$.

The capacitor $C = f^{-1}B(f(x), R) \setminus \overline{B}(x, \varepsilon)$ is separated by the spherical capacitor $S = B(x, D) \setminus \overline{B}(x, \varepsilon)$, hence such that

$$\text{capacity}\, C \leqslant \text{capacity}\, S = \tau\left(\frac{\varepsilon}{D}\right).$$

Its image $f(D)$ separates the plates of the spherical capacitor $S' = B(f(x), R) \setminus \overline{B}(f(x), r)$, whence

$$\text{capacity}\, f(C) \geqslant \text{capacity}\, S' = \tau\left(\frac{r}{R}\right).$$

The function τ is nondecreasing, so $r/R \leqslant \varepsilon/D$, and then

$$\mathrm{Lip}_f(x) \leqslant \frac{R}{D(x)}.$$

Since D is continuous, the local dilation is bounded. Especially, f is (uniformly) Lipschitz on every straight curve on which it is almost everywhere continuous. As f is absolutely continuous along almost every curve, f is locally Lipschitz. □

When f is a global homeomorphism between Carnot groups, we can let the radius R tend to $+\infty$. We find that f is globally Lipschitz, with $\mathrm{Lip}(f) \leqslant \lim_\infty R/D$, which is in a way the local dilation of f at infinity. In particular, if the "differential of f at infinity" is an isometry, f is an isometry. This has a precise meaning on the sphere at infinity of the symmetric space.

11.5. **Proposition.** *Let X be a rank one symmetric space, let ∂X be its sphere at infinity, endowed with the conformal structure defined in §9.6. Every 1-quasiconformal global transformation of ∂X comes from an isometry of X.*

Fix two points x and ∞ in ∂X, where f is differentiable at ∞. As $\mathrm{Isom}(X)$ is 2-transitive on ∂X, we can assume without loss of generality that f fixes x and ∞. Consider two embeddings ι_x, ι_∞ of the abstract group N in $\mathrm{Isom}(X)$, that fix x and ∞ respectively. They allow us to identify $\partial X \setminus \{x\}$ and $\partial X \setminus \{\infty\}$ to N through

$$i_x(g) = \iota_x(g) \cdot \infty, \qquad i_\infty(g) = \iota_\infty(g) \cdot x.$$

Let us denote $\{A_t\}$ the one-parameter group of translations along the geodesic from x to ∞; then,

$$i_x \circ e^{t\alpha} = A_t \circ i_x, \quad i_\infty \circ e^{-t\alpha} = A_t \circ i_\infty,$$

so that

$$Df(\infty) = \lim_{t\to+\infty} A_{-t} \circ f \circ A_t.$$

Note that any potential differential, i.e., any element $\beta \in Z_{\mathrm{Aut}\,N}(\alpha)$ which is a similarity, is indeed the differential at ∞ of an isometry of X, namely it is the differential of $i_\infty \circ \beta \circ i_\infty^{-1}$. We can therefore, up to composing f with isometries of X, assume that $Df(\infty)$ is the identity. Letting $\overline{f} = i_\infty^{-1} \circ f \circ i_\infty$, $e^{-t\alpha} \circ \overline{f} \circ e^{t\alpha}$ tends to the identity when t tends to $+\infty$.

Let us prove that $\overline{f}$ is an isometry of the distance d_∞ defined on N that was introduced in 9.5. Set $R = e^t$. When y is the neutral element of N, the number $D(y)$ that was introduced in Lemma 11.4 is such that

$$\begin{aligned}\frac{D(y)}{R} &= \frac{1}{R} d_\infty \left(y, \overline{f}^{-1} \partial B(\overline{f}(y), R) \right) \\ &= d_\infty(y, e^{-t\alpha} \circ \overline{f} \circ e^{t\alpha} \partial B(\overline{f}(y), 1))\end{aligned}$$

which tends to 1 when R tends to $+\infty$. Lemma 11.4 allows one to conclude that when $y = 0$, for all $z \in Y$,

$$d_\infty \left(\overline{f}(y), \overline{f}(z) \right) \leqslant d_\infty(y, z).$$

By composing f with translations (which does not change its behavior at infinity), we obtained the same inequality for any y. Arguing on f^{-1}, we conclude that f is an isometry of (N, d_∞).

It remains to verify that an isometry of d_∞, which fixes the neutral element is group automorphism. Let us call *line* in N a horizontal curve whose projection in $N/[N,N] = V^1$ is an affine line. Note that lines are characterized by a metric property: they are the only curves that realize the distance between any two of their points. An isometry permutes the lines. Let $z\Sigma$ be the union of the lines passing through x. The elements of the center $[N,N]$ are characterized by the following property: $z \in [N,N]$ if and only if $z\Sigma \cap \Sigma = \emptyset$. Since $f(z\Sigma) = f(z)\Sigma$, f preserves the center. More generally, f permutes the fibers of the projection $T : N + N/[N,N]$, thus defining a bijection β of the vector space $V^1 = N/[N,N]$. This map preserves the Euclidean distance $\overline{d}(u,v) = d_\infty(\pi_{-1}(u), \pi_{-1}(v))$. As β commutes with the homotheties of V^1 , β coincides with the restriction to V^1 of the differential of f, whenever it exists. This proves that β extends into an automorphism of N and that, wherever it exists, the differential of $\beta^{-1} \circ f$ is the identity. With the property of absolute continuity on curves, we conclude easily that $\beta^{-1} \circ f$ is itself the identity. $\square$

11.6. **Proof of Theorem 1.** Let f be a quasiisomety of quaternionic hyperbolic space $\mathbf{H}\mathrm{H}^n$ or the hyperbolic Cayley plane $\mathbf{Ca}\mathrm{H}^2$. By §9.12,

f extends into a quasiconformal homeomorphism of the sphere at infinity. The latter can be identified withh a Carnot group (9.3). The homeomorphism f is automatically 1-quasiconformal, so there exists an isometry $\widetilde{f}$ which has the same extension to the sphere at infinity (11.5). Finally, by 9.11, f is a bounded distance from $\widetilde{f}$, i.e. $d(f(x), \widetilde{f}(x))$ is bounded above and the bound depends on the quasiisometry constants L and C.

11.7. **Proposition.** (Tukia, [Tuk85]). *Every quasiconformal transformation of ∂X extends in a quasiisometry of $X = \mathbf{R}\mathrm{H}^n$.*

I do not know how to extend this to the complex hyperbolic spaces. However, it is clear that a contact transformation with compact support (and, more generally, a biLipschitz homeomorphism relative to a Carnot-Caratheodory metric) of a complex Heisenberg group extends into a extends into a quasi-isometry of complex hyperbolic space.

In order to show this, let us consider the coordinates (p, t) on $\mathbf{C}\mathrm{H}^n = N \times \mathbf{R}$. The Riemannian metric is $g_t \oplus dt^2$ where $e^{-2t} g_t = g_0 + e^{2t}\omega^2$ and ω is a contact form. If f is a contact homeomorphism with bounded derivative, one has

$$f^* e^{-2t} g_t \leqslant \mathrm{const}.g_0 + e^{2t}\mathrm{const}.\omega^2 \leqslant \mathrm{const}.e^{-2t} g_t,$$

so that f is uniformly bilipschitz with respect to the metrics g_t. Setting $\widetilde{f}(p,t) = (f(p), t$ we obtain a quasiisometry which extends as f to the sphere at infinity.

We have emphasized a specific property of quaternionic and Cayley hyperbolic spaces. However, it is clear that once the conformal structure of the sphere at infinity is understood, one can easily generalize some some properties of real hyperbolic spaces to all rank one symmetric spaces. For example, here[25] is a generalization of a result of M. Gromov [Gro81a] and P. Tukia [Tuk85]: a discrete group of finite type which is quasi-isometric to a simple Lie group G of rank one is commensurable to a discrete cocompact subgroup of G.

Part C. Negatively curved homogeneous spaces

We extend to a large class of homogeneous space with negative sectional curvature the results of part B. Theorem 2 then allows to classify these spaces up to quasiisometry. In addition, we prove that the rigidity property that was seen in Theorem 1 is shared by some of these spaces.

12. Carnot type homogeneous spaces

It follows from the work of E. Heintze that any homogeneous Riemannian manifold of negative curvature can be seen as a left-invariant metric on a solvable Lie group R of the following type: R is a semidirect product $N \rtimes_\alpha \mathbf{R}$, where N is a nilpotent Lie group and $\mathbf{R}$ acts on the Lie algebra $\mathcal{N}$ by a derivation α whose eigenvalues all have positive real parts [Hei74]. The metric on R can be written

$$g = g_t \oplus dt^2,$$

where

$$g_t = \left(r^{t\alpha}\right)^* g_0$$

and g_0 is a left-invariant Riemannian metric on N.

Conversely, if N is a nilpotent Lie group and if α is a derivation whose eigenvalues have positive real parts, the extension $N \rtimes_\alpha \mathbf{R}$ admits a left-invariant metric with negative sectional curvature. This class contains the Carnot type groups defined in 1.2, and especially, the symmetric spaces of rank one, seen as invariant metrics on the minimal parabolic AN.

We shall see that the considerations of part B apply without change to the homogeneous spaces of Carnot type, that is, those such that N and α are as in Definition 1.2.

12.1. **Sphere at infinity.** For every $p \in N$, the curve $s \mapsto (p, s)$ is a geodesic of R, and

$$\begin{aligned} d((p,t),(q,t)) &\leqslant d_{g_t}(p,q) \\ &= d_{g_0}(e^{t\alpha}(p), e^{t\alpha}(q)) \end{aligned}$$

tends to 0 when t tends to $+\infty$. The geodesic rays $\{p = \text{const.}, t > 0\}$ define the same point at infinity, denoted ∞. One such geodesic goes through any point of R, we have all the geodesics going to ∞.

12.2. **Metric on the sphere at infinity.** Let us endow $\partial R \setminus \{\infty\}$, identified with N, with the Carnot-Carathéodory metric d_∞ attached to the invariant plane field V^1, equipped with the restriction of g_0. R acts on N by similarities of d_∞. The proof of Lemma 9.12 applies.

12.3. **Lemma.** *A quasiisometry between two Carnot type homogeneous space extends to a quasiconformal homeomorphism between their spheres at infinity. This homeomorphism is locally quasiconformal with respect to the Carnot-Carathéodory metrics d_∞.*

12.4. **Corollary.** *If two solvable groups of Carnot type are quasiisometric, then they are isomorphic.*

Indeed, by Theorem 2, there is an isomorphism $N \to N'$ that intertwines the derivations α and α'. This isomorphism extends into an isomorphism between the Carnot type groups $N \rtimes_\alpha \mathbf{R}$ and $N \rtimes_{\alpha'} \mathbf{R}$.

More generally, the classification of homogeneous spaces with negative sectional curvature up to quasiisometry was solved by U. Hamenstädt, [Ham87].

13. Graded automorphism groups of certain Carnot groups

In §10 we proved that the Lie algebras $\mathcal{N}$ attached to the quaternionic and Cayley hyperbolic spaces have the following property: the group of graded automorphisms of $\mathcal{N}$ is the direct product of R with a compact group. This property is shared with numerous other Carnot Lie algebras, that is, Lie algebras $\mathcal{N}$ endowed with a gradation

$$\mathcal{N} = V^1 \oplus \cdots \oplus V^r,$$

such that V^1 generates $\mathcal{N}$.

The following examples were communicated to me by Y. Benoist [Ben]. Let $\mathcal{G}$ be a compact simple Lie algebra, let $\mathcal{N}$ be the vector space of degree d polynomials, without constant terms, in one variable T, with coefficients in $\mathcal{G}$. The Lie algebra $\mathcal{G}$ induces a structure of a Lie algebra on $\mathcal{N}$, that is nilpotent of rank d. Y. Benoist shows that, when $\mathcal{G}$ has no ideal that is isomorphic to $\mathfrak{sl}(2,\mathbf{C})$ or one of its real forms, the group of graded automorphisms of $\mathcal{N}$ decomposes as $G_a \times \mathbf{R}$, where $G_a = \mathrm{Aut}(\mathcal{G})$ is the adjoint group of $\mathcal{G}$.

There even exist graded Lie algebra whose only graded automorphisms are homotheties. The filiform graded Lie algebras (i.e. those for which the dimensions of the subspaces in the gradation are the least possible, namely $2, 1, \ldots, 1$) were classified by M. Vergne [Ver68]. In even[26] dimension there are two such algebras, one has unipotent graded automorphisms, the other has only homotheties.

In the following discussion, we establish that this property holds generically for a class 2 Lie algebra which has $\dim V^1 = n$ and $\dim V^2 = p$ with $2 < p < 2n - 3$. A Carnot Lie algebra of class 2 is completely described by the vector spaces V^1 and V^2 together with an alternate bilinear surjective map $[\cdot,\cdot]\colon \Lambda^2 V^1 \to V^2$. An isomorphism between two such algebras is given by a pair of isomorphisms $A \in \mathrm{Gl}(V^1)$ and

$B \in \mathrm{Gl}(V^2)$ such that

$$[\cdot,\cdot]' \circ \Lambda^2 A = B \circ [\cdot,\cdot].$$

The choice of $[\cdot,\cdot]$ amounts to the choice of a map from $(V^2)^*$ to $\Lambda^2(V^1)^*$. Let us denote by Z the image of this map. In this way we have a bijection between the classes of structures $[\cdot,\cdot]$ up to isomorphisms and linear subspaces of $\Lambda^2(V^1)^*$ up to $\mathrm{Gl}(V^1)$. The space of Carnot Lie algebras of class 2 with $\dim V^1 = n$ and $\dim V^2 = p$ has the structire of a quotient of an algebraic variety by a group acting algebraically, and one can speak of a generic property: this means a property that holds in a Zariski open subset of the Grassmannian of p-planes in $\Lambda^2\mathbf{R}^n$.

13.1. **Proposition.** *Let is consider the Lie algebra of Carnot type, of class 2, with dimension n an p such that*

$$n \textit{ is even}, \ n \geqslant 10, \ 3 \leqslant p \leqslant 2n-3.$$

Generically, all the automorphisms of such an algebra are homotheties.

The proof requires three steps:

(1) Let $(A,B) \in \mathrm{Gl}(n,\mathbf{R}) \times \mathrm{Gl}(p,\mathbf{R})$ be an automorphism. By definition of a Carnot Lie algebra, A uniquely determinates B. We prove that generically, B detemines A. This comes from the fact that, again generically, three symplectic forms share no automorphism.
(2) The subgroup $\mathrm{Aut}_\alpha(\mathcal{N}) \subset \mathrm{Gl}(p,\mathbf{R})$ preserves the Pfaffian. The latter being an irreducible polynomial of degree $n/2$, this implies that this group is finite when $p < 2n-3$.
(3) If the Lie algebra defined by a subspace Z of $\Lambda^2(\mathbf{R}^n)^*$ admits an automorphism of finite order, then Z is very special.

Step 1. Let $A \in \mathrm{Gl}(n,\mathbf{R})$ be such that $(A,1)$ is a graded automorphism of $\mathcal{N}$, i.e.

$$[Ax,Ay] = [x,y] \text{ for all } x,y \in \mathbf{R}^n.$$

Generically, at least one of the components of $[\cdot,\cdot]$ is a symplectic form, that we denote ω. The others can be written as

$$\omega'(x,y) = \omega(Cx,y), \quad \omega''(Dx,y),$$

for certain $C,D \in \mathrm{End}(\mathbf{R}^n)$. Since A preserves ω, ω' and ω'', it commutes with C and D. If $p \geqslant 3$, for generic C and D, only the homotheties commute with C and D. We can conclude that A is a homothety, hence $A=1$. This proves that generically, $\mathrm{Aut}_\alpha(\mathcal{N})$ embeds in $\mathrm{Gl}(p,\mathbf{R})$.

Step 2. On $\Lambda^2(\mathbf{R}^n)^*$, the Pfaffian is defined. It is a $\mathrm{Gl}(n,\mathbf{R})$-invariant polynomial of degree $n/2$. It is irreducible, and vanishes exactly on the algebraic subset Σ of degenerate 2-forms. By Darboux's theorem[27] there are $n/2$ orbits of $\mathrm{Gl}(n,\mathbf{R})$ in $\Lambda^2(\mathbf{R}^n)^*$. The orbit O_r of two-forms with rank r has dimension

$$\dim O_r = n^2 - \frac{2r(2r+1)}{2} - \frac{(n-2r)^2}{2}.$$

It follows that the hypersurface Σ has a singular locus of codimension at least $2n-3$. If $p \leqslant 2n-3$, a generic p-plane Z in $\Lambda^2(\mathbf{R}^n)^*$ cuts Σ along a sommoth, irreducible hypersurface Σ_{Z}. Let us complexify the situation. In $\mathbf{P}(\mathrm{Z}^{\mathbf{C}})$, the group $\mathrm{Aut}_\alpha(\mathcal{N})$ stabilizes $\Sigma_{\mathrm{Z}^{\mathbf{C}}}$. This hypersurface is smooth, irreducible, of degree $n/2 > p$, hence its first Chern class is negative. Especially its automorphism group is discrete, hence it is finite[28]. We can conclude that $\mathrm{Aut}_\alpha(\mathcal{N})$ is finite if $p < n/2$. In fact we can push until $p = 2n-3$; I owe this observation to O. Debarre. Let us denote by P the Pfaffian restricted to Z.

If Z is smooth, $d, \partial f/\partial x_1, \ldots, \partial f/\partial x_p$ is a regular sequence of functions. It then follows from the acyclicity of Koszul's complex (See [GH78, page 688] for a local version of the result we need) that the equation

$$\sum_{i=1}^{l} l_i \frac{\partial P}{\partial x_i} = 0, \qquad l_i \in \mathrm{Z}^*$$

has no non-zero solution, i.e., the group of linear preserving P is still finite.

Step 3. Assume that $\mathcal{N}$ has a graded automorphism (A,B) with finite order l. Let ζ be a primitive l-th root of unity, and denote

$$E_i(A) = \{x \in \mathbf{C}^n \colon Ax = \zeta^i x\}.$$

The eigenvalues of $\Lambda^2 A$ are again l-rooth of unity and

$$E_k(\Lambda^2 A) = \sum_{i \leqslant j, i+j = k \bmod l} E_i(A) \otimes E_j(A)$$

where we use the covention $E_i(A) \otimes E_i(A) = \Lambda^2 E_i(A)$. If Z is a A-invariant p-subspace of $\Lambda^2\mathbf{R}^n$, it is a direct sum of its intersection with the eigenspaces of $\Lambda^2 A$, i.e.,

$$\mathrm{Z} = \sum_{k=1}^{l} \mathrm{Z}_k$$

where

$$\mathrm{Z}_k = Z \cap E_k(\Lambda^2 A).$$

Define $n_i = \dim E_i(A)$ and $p_k = \dim \mathrm{Z}_k$. When A is fixed, the dimension of the A-invariant p-subspaces is less or equal than

$$\sum_{k=1}^{l} p_k \left(\sum_{i\leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right)$$

where

$$n_i \cdot n_j = \begin{cases} n_i n_j & i \neq j \\ \frac{n_i(n_i-1)}{2} & i = j \end{cases}$$

The algebraic set of possible A has a dimension that is equal to

$$n^2 - \sum_{i=1}^{l} n_i^2.$$

Consequently, the set of p-subspaces Z in $\Lambda^2\mathbf{R}^n$ which admit an automorphism of order l has dimension at most

$$\mu(l) = \max \left\{ n^2 - \sum_{i=1}^{l} n_i^2 + \sum_{k=1}^{l} p_k \left(\sum_{i\leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right) \right\}$$

the maximum being taken over the pairs of partitions n_i of n and p_k of p such that at least two of the n_i are nonzero.pIf $\mu(l) < p(n(n-1)/2 - P)$ one can conclude that a generic p-subspace of $\Lambda^2\mathbf{R}^n$ has no automorphism of order l. In the next Scholia, we prove that this is the case for every value of l when p and n are taken as in the statement of Proposition 13.1.

13.2. **Scholia.** *Let $n \geqslant 10$, $p \geqslant 3$, $p \leqslant n(n-1)/2 - 3$. For every pair of partitions $\sum_{i=1}^{l} n_i = n$ and $\sum_{k=1}^{l} p_k = p$ such that for all k,*

$$\sum_{i\leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \geqslant 0,$$

it holds

$$n^2 - \sum_{i=1}^{l} n_i^2 + \sum_{k=1}^{l} p_k \left(\sum_{i\leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right) \leqslant p \left(\frac{n(n-1)}{2} - p \right)$$

and equality occurs only if all but one of the n_i vanish.

Note the symmetric roles of p_k and of $\sum_{i\leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k$ which allows to assume without loss of generality that $p \leqslant n(n-1)/4$.

Denote $x = \max\{p_k\}$. Obviously,

$$\begin{aligned} X &= \sum_{k=1}^{l} p_k \left(\sum_{i \leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right) + n^2 + \sum_{i=1}^{l} n_i^2 \\ &\leqslant x \left(\sum_{k=1}^{l} \sum_{i \leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right) + n^2 + \sum_{i=1}^{l} n_i^2 \\ &= x \left(\frac{n(n-1)}{2} - p \right) + n^2 + \sum_{i=1}^{l} n_i^2, \end{aligned}$$

hence[29]

$$\text{(I)} \qquad \begin{aligned} Y &= p \left(\frac{n(n-1)}{2} - p \right) - X \\ &\geqslant a = (p-x) \left(\frac{n(n-1)}{2} - p \right) - n^2 + \sum_{i=1}^{l} n_i^2. \end{aligned}$$

Furthermore,

$$\sum_{i<j, i+j=k \bmod l} n_i n_j \leqslant \frac{1}{2} \sum_{i \neq j, i+j=k \bmod l} \frac{n_i^2 + n_j^2}{2} = \frac{1}{2} \sum_{2i \neq k \bmod l} n_i^2,$$

so that

$$\sum_{i \leqslant j, i+j=k \bmod l} n_i \cdot n_j \leqslant \frac{1}{2} \sum_{i=1}^{l} n_i^2$$

and then,

$$\begin{aligned} X &= n^2 - \sum_{i=1}^{l} n_i^2 + \sum_{k=1}^{l} p_k \left(\sum_{i \leqslant j, i+j=k \bmod l} n_i \cdot n_j - p_k \right) \\ &\leqslant \frac{p-2}{2} \sum_{i=1}^{l} n_i^2 + n^2 - \sum_{k=1}^{l} p_k^2, \end{aligned}$$

that is,

$$\text{(II)} \qquad \begin{aligned} Y &= p \left(\frac{n(n-1)}{2} - p \right) - X \\ &\geqslant b = \frac{p-2}{2} \left(n^2 - \sum_{i=1}^{l} n_i^2 \right) - \frac{pn}{2} - p^2 + x^2. \end{aligned}$$

Combining (I) and (II) yields

$$\begin{aligned}\frac{p}{2}Y &\geqslant \frac{p-2}{2}a+b\\ &\geqslant f(x)=\frac{p-2}{2}(p-x)\left(\frac{n(n-1)}{2}-p\right)-\frac{np}{2}-p^2+x^2.\end{aligned}$$

Note that, as $p\leqslant n(n-1)/4$, the function f attains its minimum value at

$$x_0=\frac{p-2}{4}\left(\frac{n(n-1)}{2}-p\right)$$

which is greater or equal to p. It follows that f is nonincreasing on $[0,p]$. Especially, for $x\leqslant p-1$ one has

$$f(x)\geqslant f(p-1)=g(p)=-\frac{1}{2}$$

The function g is concave, attains its maximal value at $p=n(n-1)/4-n+2/4$, which is very close to $n(n-1)/4$. At $p=3$ its value is

$$g(3)=\frac{1}{4}(n^2-7n-26)$$

which is positive as soon as $n\geqslant 10$. We thus proved that $Y>0$ when $x\leqslant p-1$.

One case remains, that of $x=p$, which occurs when only one of the p_k is nonzero and equal to p.

We need to prove that

$$n^2-\sum_{i=1}^{l}n_i^2+p\left(\sum_{i\leqslant j,\, i+j=k \,\mathrm{mod}\, l}n_i\cdot n_j\right)<\frac{pn(n-1)}{2},$$

or otherwise said, that

$$n^2-\sum_{i=1}^{l}n_i^2+\frac{p}{2}\left(\sum_{i=1}^{l}n_in_{k-i}-\sum_{2i=k \,\mathrm{mod}\, l}n_i\right)<\frac{pn(n-1)}{2}.$$

Note that the left-hand side can only increase if, while keeping n_i+n_{k-i} fixed, we bring n_i closer to n_{k-i}. We can thus assume that $n_i=n_{k-i}$ for all $i=1,\ldots,l$. And the inequality that we need to prove then becomes

$$n^2-\sum_{i=1}^{l}n_i^2>\frac{p}{p-2}\left(\sum_{2i\neq k \mod l}n_i\right).$$

Let us distinguish between three cases.

First case. We can group the n_i in two collections with sums c and d such that $c \geqslant 2$ and $d \geqslant 2$. Then

$$n^2 - \sum_{i=1}^{l} n_i^2 = \sum_{i \neq j} n_i n_j \geqslant 2cd \geqslant 4n - 8 > 3n \geqslant \frac{p}{p-2} \left(\sum_{2i \neq k \mod l} n_i \right),$$

where the last inequality holds because $n \geqslant 10$, $p \geqslant 3$ and $\sum_{2i \neq k \mod l} n_i \leqslant n$.

Second case. All the n_i are equal to 0 or 1. Then

$$n^2 - \sum_{i=1}^{l} n_i^2 = n^2 - n > 3n \geqslant \frac{p}{p-2} \left(\sum_{2i \neq k \mod l} n_i \right).$$

Third case. One of the n_i is equal to $n-1$, the other are equal to 0. Then we go back to the previous expression: in the worst case,

$$\sum_{i \leqslant j, i+j=k \mod l} n_i \cdot n_j \leqslant \frac{(n-1)(n-2)}{2},$$

and we have

$$2n + p\left(\frac{(n-1)(n-2)}{2} - p\right) < p\left(\frac{n(n-1)}{2} - p\right). \qquad \square$$

13.3. **Remark.** By comparing the dimension of $\mathrm{Gl}(n, \mathbf{R})$ with that of the algebraic set of 2-planes of $\Lambda^2\mathbf{R}^n$, we see that all Carnot Lie algebras with either $p \leqslant 2$, $p \geqslant n(n-1)/2 - 2$ or $n \leqslant 4$ have a non-trivial group of graded automorphisms. On the other hand, it is likely that the result of Proposition 13.1 holds when $5 \leqslant n \leqslant 9$.

14. Quasiisometries of certain homogeneous spaces of Carnot type

14.1. **Definition.** A solvable Lie group R belongs to the class (C) if the following holds:

- If $N = [R, R]$ denotes the commutator subgroup, then $\operatorname{codim} N = 1$.
- There exists $r \in R \setminus N$ such that N equipped with the inner automorphism r is a class 2 Carnot group.
- The centraliser of r in $\mathrm{Aut}(N)$ is the one-parameter subgroup going through r.

Theorem 4. *Let R be a solvable group of class (C), endowed with a left-invariant Riemannian distance. Any quasiisometry of R is a bounded distance away from an inner automorphism.*

By [Hei74] we can assume thet the invariant metric has negative sectional curvature. Lemmas 12.3 and 9.11 bring us back to showing that any quasiconformal transformation of the sphere at infinity $\partial R = N \cup \infty$ which, for example, fixes the origin of N, is a homothety, i.e., in the one-parameter group generated by r. Let f be such a homeomorphism. It admits almost everywhere a differential which is an automorphism of N commuting with r, so a power[30] of r. For almost any line $y(t) = x \exp(tv)$, where $v \in V^1$, the curve $f \circ \gamma$ is absolutely continuous, differentiable almost everywhere. If π denotes the projection of N onto $N/[N,N] = V^1$, then for almost every t,

$$D(\pi \circ f \circ \gamma) = D\pi \circ Df(\gamma(t))(v)$$

is colinear with v. As $\pi \circ f \circ \gamma$ is absolutely continuous, this implies that $\pi \circ f \circ \gamma$ is contained in a line of V^1, and that its multiplicative integral $f \circ \gamma$ is contained in a line of N. By continuity, f sends any line in a line.

Let $u, v, w \in V^1$. Consider the set of closed "quadrilaterals" with sides parallel to $u, v, -u, w$, i.e, the 4-uples of points in N of the form

$$(x, x\exp(qu)\exp(rv), x\exp(qu)\exp(rv)\exp(su),\\ x\exp(qu)\exp(rv)\exp(su)\exp(tw)),$$

where $q, r, s, t \in \mathbf{R}$ are such that

$$x\exp(qu)\exp(rv)\exp(su)\exp(tw) = x.$$

Th set of 4-tuples of points as above is invariant under f. For a group of class 2, we can compute that

$$\begin{aligned} &x\exp(qu)\exp(rv)\exp(su)\exp(tw)) \\ &= x\exp\left(qu + rv + su + tw + \frac{1}{2}(r(q-s)[u,v] + [qu + rv + su, tw])\right). \end{aligned}$$

If $[u, v] \neq 0$, i.e., if the quadrilateral, once translated with the vertex x at the origin, is not contained in an abelian subgroup of N, the quadrilateral can close only if $q = s$. So we proved that if two line segments of the form $(x, x\exp(qu))$ can be placed in a same non-abelian quadrilateral, then they have equal length, and their images through f have equal length.

By assumption, the group N does not decomposes as a direct product. We can thus select $u \in V^1$ such that the open subset

$$U = \{v \in V^1 : [u, v] \neq 0$$

is non-empty. For $x \in N$, set

$$a(x) = d(f(x), f(x \exp(u))).$$

If $v \in U$, then

$$a(x \exp(u) \exp(v)) = a(x)$$

since $(x, x\exp(u); x\exp(u)\exp(v), x\exp(2u+v), x)$ is a closed non-abelian quadrilateral. By continuity, this remains true for all $v \in V^1$. We can check easily that any two points in N can be joined by a chain of points of the form $x_{i+1} = x_i \exp(u) \exp(v_i)$, so the function $a = a_u$ such that the identity holds is a constant. Consequently, the function

$$\lim_{s \to 0} \frac{1}{s} a_{su}(x) = \mathrm{Lip}_f(x)$$

is constant as well. For $t = \mathrm{Lip}_f$, the derivative of $e^{-t\alpha} \circ f$ is almost everywhere equal to the identity map, so f is the product of a left translation and a homothety $e^{t\alpha}$; the property of absolute continuity along almost every line intervenes here. ☐

14.2. **Remark.** We saw in §13.1 that the class (C) contains many extensions of class 2 groups.

Conversely, if N is a Carnot group which has a graded automorphism ϕ that generates a non relatively compact subgroup of $\mathrm{Aut}_\alpha(N)/\{e^{t\alpha}\}$, then ϕ gives a quasi-isometry of the group $R = N \rtimes_\alpha \mathbf{R}$ which is not close to any isometry.

14.3. **An example of a homogeneous space whose quasiisometry group has infinite dimension.** Let M be a differentiable manifold. Denote $\pi\colon PM = \mathbf{P}(TM) \to M$ the projectivized tangent bundle. There is a natural plane field Q on this bundle: at $(x, d) \in PM$,

$$Q = \{v \in TPM \colon \pi_* v \in d\}$$

We claim that this plane field is accessible. Indeed, if $c(s)$ is a curve in M, the curve $(c(s), \dot{c}(s))$ is tangent to Q. One can always find a curve passing at prescribed pair of point with a prescribed pair of tangents at these points. Hence the curves that are tangent to Q can join any two points of PM. Every diffeomorphism of M naturally lifts as a diffeomorphism of PM which preserves Q.

Now set $M = \mathbf{R}^{n+1}$ with coordinates $p_0, \dots, p_n$. A line that is transversal to the hyperplane of equation $\dot{p}_0 = 0$ can be charted by the components of a direction vector $(1, q_1, \dots, q_n)$. This provides coordinates in the open set $N = \{\dot{p}_0 \neq 0\}$ of $P\mathbf{R}^{n+1}$. On this open set the vector fields

$$Z_i = \frac{\partial}{\partial p_i}, \quad X_i = \frac{\partial}{\partial q_i}, \quad 1 \leqslant i \leqslant n, \quad Y = \frac{\partial}{\partial p_0} + \sum_{i=1}^{n} q_i \frac{\partial}{\partial p_i}$$

are such that

$$X_i, X_j] = [X_i, Z_j] = [Y, Z_j] = 0, \ [X_i, Y] = Z_i.$$

It follows that the open set N has the structure of a nilpotent Lie group, and the plane field Q generated by Y and the X_i is left-invariant. Once endowed with the derivation

$$\alpha(X_i) = X_i, \quad \alpha(Y) = Y, \quad \alpha(Z_i) = 2Z_i,$$

N is a Carnot group.

A diffeomorphism of $M = \mathbf{R}^{n+1}$ preserves N if and only if it preserves the foliation by hyperplanes $\{p_0 = \text{const.}\}$. This provides many quasiconformal homeomorphisms (in fact, these are bilipschitz homeomorphisms) of the Carnot group N, and then many quasiisometries of the homogeneous space $N \rtimes_\alpha \mathbf{R}$.

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Translator's note: Some of the original references have been updated and we refer to their English translation when there is one.

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Notes

1 Namely, $\mathrm{O}(n,1)$ and $\mathrm{U}(n,1)$ on one side, $\mathrm{Sp}(n,1)$ and F_4^{-20} on the other.

2 The translator has chosen to translate systematically the author's "Carnot" with "Carnot type" when it comes to designate the negatively curved homogeneous spaces with a Carnot group at infinity. This is the modern terminology, it already appears sporadically in the paper, and allows to avoid any confusion with the Carnot group themselves.

3 In trying to be more specific and in view of Theorem 4 one may at first expect that the real and complex hyperbolic spaces are the only negatively curved homogeneous spaces with an infinite-dimensional group of quasiisometries. This is, however, not true. While some degree of rigidity has been proved for the quasiisometries of certain non-symmetric homogeneous spaces beyond the one considered in this paper, this rigidity is not always as strong as the one observed for $\mathrm{Sp}(n,1)$ and F_4^{-20}, as already indicated by the examples in §14.3. An important case is that of Carnot type spaces over filiform groups, alluded to in §13. For these, Xie has characterized the quasiconformal homeomorphisms of their boundaries as the bilipschitz maps [*Michigan Math. J.* **64** (2015), no. 1, 169–202]. They do form an infinite-dimensional space.

4 Nowadays, thanks to the work of Heinonen and Koskela there is a well-founded theory of quasiconformal homeomorphisms between metric spaces. One should beware that what the author calls a "quasiconformal homeomorphism" in this paper is in general a slightly different notion, and corresponds to "(uniformly locally) quasisymmetric homeomorphism" according to the modern terminology. This was clarified in [J. Tyson, *Ann. Acad. Sci. Fenn. Math.* **23** (1998), no. 2, 525–548]. Nevertheless, between Carnot-Carathéodory metrics, quasiconformal (in the standard sense) and quasisymmetric homeomorphisms are the same.

5 Here the technique alluded to is Cartan's equivalence method.

6 Bellaiche gave a new proof of Mitchell's theorem in [*Sub-Riemannian geometry*, 1–78, Progr. Math., 144, Birkhäuser, Basel, 1996].

7 This was done subsequently, in [G. Margulis and G. D. Mostow, *Geom. Funct. Anal.* **5** (1995), no. 2, 402-433]. The proof given there is different even in the special case of Carnot groups.

8 The modern terminology is "asymptotic cone".

9 This is slightly incorrect: all nilpotent groups up to dimension 4 admit Carnot gradings, but there exists a nilpotent group of dimension 5 witout any Carnot grading. Hence the number 5 should be replaced by 4 and 6 with 5 in the next sentence.

10 What was probably meant here is that $B(xe^{-t\alpha}\mu, \lambda) \cap F \neq \emptyset$ is a $\mathscr{H}^p$-null set.

11 If one wanted a self-contained proof of Theorem 3, this special case of Proposition 3.2 would actually be sufficient.

12 One should assume here that l is a large enough integer.

13 In $(**)$, the reader should infer that $\sigma_\Sigma(s)$ denotes the point σ_Σ obtained by applying the developing procedure with the restriction of the subdivision to $[0, s]$.

14 Between open subsets of Carnot groups.

15 The 1-quasiconformal homeomorphisms $f : N \to N'$ in this paper should not be mistaken in general with the more traditional ones for which the metric distortion function

$$H_f(x) = \limsup_{r\to 0} \frac{\sup\{d(f(x), f(y) \colon d(x,y) \leqslant r\}}{\inf\{d(f(x), f(y) \colon d(x,y) \geqslant r\}}$$

is 1 at all $x \in N$.

16 Definition 7.10 makes 1-quasiconformal homeomorphisms here rather close in spirit to 1-quasiregular homeomorphisms, which originally denote $W^{1,n}_{\text{loc.}}$ homeomorphisms between open subsets of $\mathbf{R}^n$ whose derivative is almost everywhere a homothety. These were notably used by Bojarskii and Iwaniec to give a new proof of a theorem of Liouville [*Math. Nachr.* **107**, 253-262 (1982)], which is close to the strategy of the present paper in the way that it provides a striking improvement from 1-quasiregular to Möbius.

17 The function v is assumed to be locally Lipschitz.

18 In the outward direction.

19 Instead of Lemma 6.2, the reference must be to Lemma 9.4. Also, the t in the exponent should probably have no minus sign.

20 One should understand that they converge uniformly on any compact subset. If this was understood globally then the boundary maps would be equal anyway.

21 Or the union.

22 Bourdon gave a different, arguably simpler proof of Proposition 10.1 as Proposition 7.4 in [Mostow type rigidity Theorems, *Handbook of Group Actions* (Vol. IV) ALM41, Ch. 4, pp. 139–188].

23 What is meant here is

$$\otimes_0^2 \mathbf{R}^4 = S_0^2 \mathbf{R}^4 \oplus \Lambda^2 \mathbf{R}^4.$$

24 This was already defined in §7.10.

25 At the time of publication, as far as symmetric spaces were concerned this property (later called quasiisometric rigidity) was only proved for real, quaternionic and Cayley hyperbolic spaces by the works of Tukia and the present paper. Later Chow gave a proof for the case of complex hyperbolic space [*Trans. Amer. Math. Soc.* 348 (1996), no. 5, 1757–1769].

26 In even dimension larger or equal to 6.

27 Only the linear version of the theorem, which pertains to linear algebra, is invoked here.

28 When the first Chern class is negative, the space of holomorphic vector fields, which is the Lie algebra of the automorphism group, is zero.

29 Equation (I) is numbered as (1) in the original, but we renamed it in order to prevent any confusion with Eq. (1) on page 28.

30 With a real exponent.

The translator thanks the author for answering all his queries.

Translation and notes by Gabriel Pallier,
Paris, July 2022.