Invariants for Sublinear Bilipschitz Equivalence

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Abstract

Sublinear biLipschitz equivalences were introduced by Cornulier in order to describe the asymptotic cones of connected Lie groups. They include and generalize quasiisometries. We classify a subclass of homogeneous spaces of negative curvature up to sublinear biLipschitz equivalence.

1. Asymptotic cones, QI and SBE

Let Y be a metric space. Informally, asymptotic cones of Y are pictures of Y taken from infinitely far away. They depend on parameters:

- A sequence of observation centers $(o_j)_{j \in \mathbf{N}}$ in Y.
- A sequence of positive scaling factors $(\sigma_j)_{j \in \mathbf{N}}$ with $\lim_j \sigma_j = +\infty$.

Given these data, $\{y_j \in Y^{\mathbf{N}} : d(y_j, o_j) = O(\sigma_j)\}$ has a semi-distance $d(y, y') = \lim_{j \to \omega} \frac{d(y_j, y_j')}{\sigma_j}$ where ω is a nonprincipal ultrafilter on $\mathbf{N}$. Denote $\mathrm{Cone}_\omega(Y, o_j, \sigma_j)$ and call asymptotic cone this space modulo the zero-distance relation.

Definition Let $f : Y \to Y'$ between metric spaces. Let $\lambda \geqslant 1$.

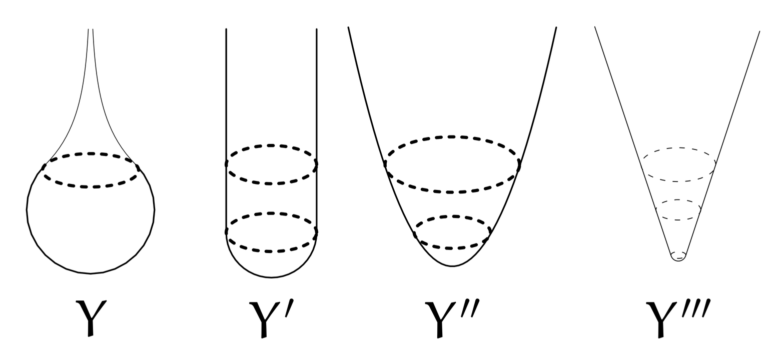
- Y is a quasiisometry with large-scale Lipschitz constant λ if for all (o_j), (σ_j) as above, for all ω, f induces a λ-biLipschitz homeomorphism
$$\mathrm{Cone}_\omega(Y, o_j, \sigma_j) \to \mathrm{Cone}_\omega(Y', f(o_j), \sigma_j).$$
- Y is a sublinear biLipschitz equivalence (SBE) if for all $o \in Y$, for all σ_j as above, for all ω, f induces a λ-biLipschitz homeomorphism
$$\mathrm{Cone}_\omega(Y, o, \sigma_j) \to \mathrm{Cone}_\omega(Y', f(o), \sigma_j).$$

Let $f : Y \to Y'$ be a SBE and $o \in Y$, denote $|y| = d(y, o)$. There exists $v(r) \ll r$ a positive real function such that for all $y_1, y_2 \in Y, y_0' \in Y'$

- $d(y_0', f(Y)) \leqslant v(d(y_0', f(o))$
- $\lambda^{-1} d(y_1, y_2) - v(|y_1| + |y_2|) \leqslant d(f(y_1), f(y_2) \leqslant \lambda d(y_1, y_2) + v(|y_1| + |y_2|)$.

Further, if f is a quasiisometry then there is c such that $v(r) < c < +\infty$. SBEs with $v = O(\log)$, resp. $v = O(r^e)$ for some $e \in [0, 1)$ can be composed.

Exercise Classify Y, Y', Y'', Y''' up to QI and SBE.



There are examples between homogeneous spaces:

★ (related to Pansu's first thesis and to [5]) Let G be a nilpotent simply connected Lie group. There exists $e \in [0, 1)$ such that G is $O(r^e)$-SBE to the Carnot group G_∞ associated to G.

▼ (Cornulier) Let A be a nonsingular $d \times d$ matrix and let D be its diagonalisable part. Then $\mathbf{R}^d \rtimes_A \mathbf{R}$ and $\mathbf{R}^d \rtimes_D \mathbf{R}$ are $O(\log)$-SBE, where $t \in \mathbf{R}$ acts by e^{tA}, resp. e^{tD}. (See Figure a)

Here are a few properties that are known to be SBE invariant for compactly generated locally compact groups: Gromov hyperbolicity, linear growth, polynomial growth, subexponential growth. Dehn functions are not invariants but this lack of invariance can be quantified, see **3**.

2.a. SBE and Gromov boundaries

Cornulier proved [3] that a SBE map $f : X \to Y$ between proper geodesic hyperbolic spaces induces $\partial_\infty f : \partial_\infty X \to \partial_\infty Y$, a homeomorphism that is bi-Hölder with respect to visual metrics.

Theorem Assume the same, and in addition that f is $O(v)$-SBE with doubing v. Then $\partial_\infty f$ distorts the asphericity of small ellipsoids by an amount sublinear in the class $O(v)$ with respect to the opposite of the logarithm of their diameter.

Conformal dimension and algebras of functions of locally bounded p-variation admit generalizations that are invariant under those mappings.

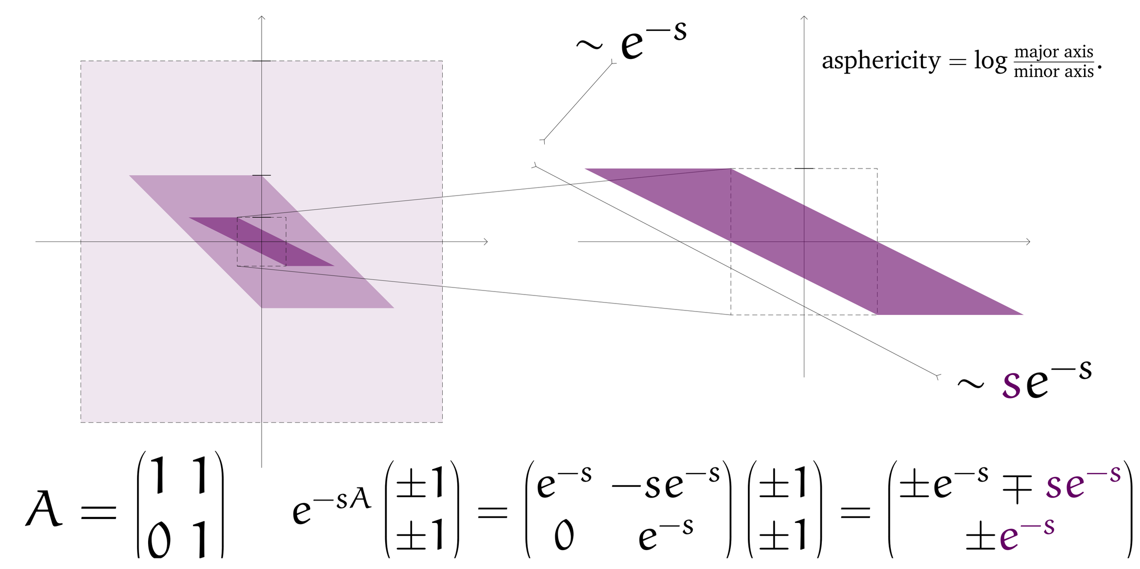


Figure a: Small balls in the visual boundary of $\mathbf{R}^2 \rtimes_A \mathbf{R}$ minus the focal point (see below). If compared to Euclidean balls at the boundary of $\mathbf{R}^2 \rtimes_D \mathbf{R}$ (or $\mathbb{H}^3_{\mathbf{R}}$), asphericity grows sublinearly with respect to the opposite of the logarithm of the diameter.

2.b. Riemannian homogeneous spaces of negative curvature

Heintze proved that a connected homogeneous manifold of negative curvature (HMN) is a left-invariant metric of a $S = N \rtimes_\alpha \mathbf{R}$ where $\alpha \in \mathrm{Der}(\mathrm{Lie}(N))$ has $\Re\lambda > 0$ for every $\lambda \in \mathrm{sp}(\alpha)$. Special cases are those S that act simply transitively on $\mathbb{H}^n_{\mathbf{K}}$ and those where α generates Carnot dilations on N (called Carnot type).

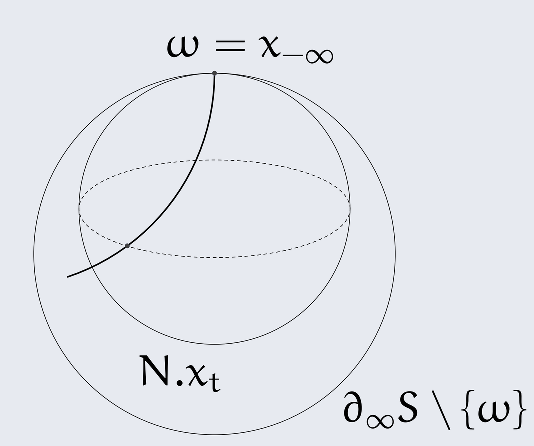


Figure b: Heintze group. N is a chart of $\partial_\infty S$ minus the focal point ω.

Some (weak) rigidity of QIs is conjectured for the Carnot type S that do not act on $\mathbb{H}^n_{\mathbf{R}}$ or $\mathbb{H}^n_{\mathbf{C}}$. Also expected:

- ⊙ Rigidity: A f.g. group QI to a HMN would be finite extension of a uniform lattice in a $\mathbb{H}^n_{\mathbf{K}}$.
- ▲ Classification: if S, S' are QI, they should be cocompact in the same Lie group (known if N, N' abelian [11]) .

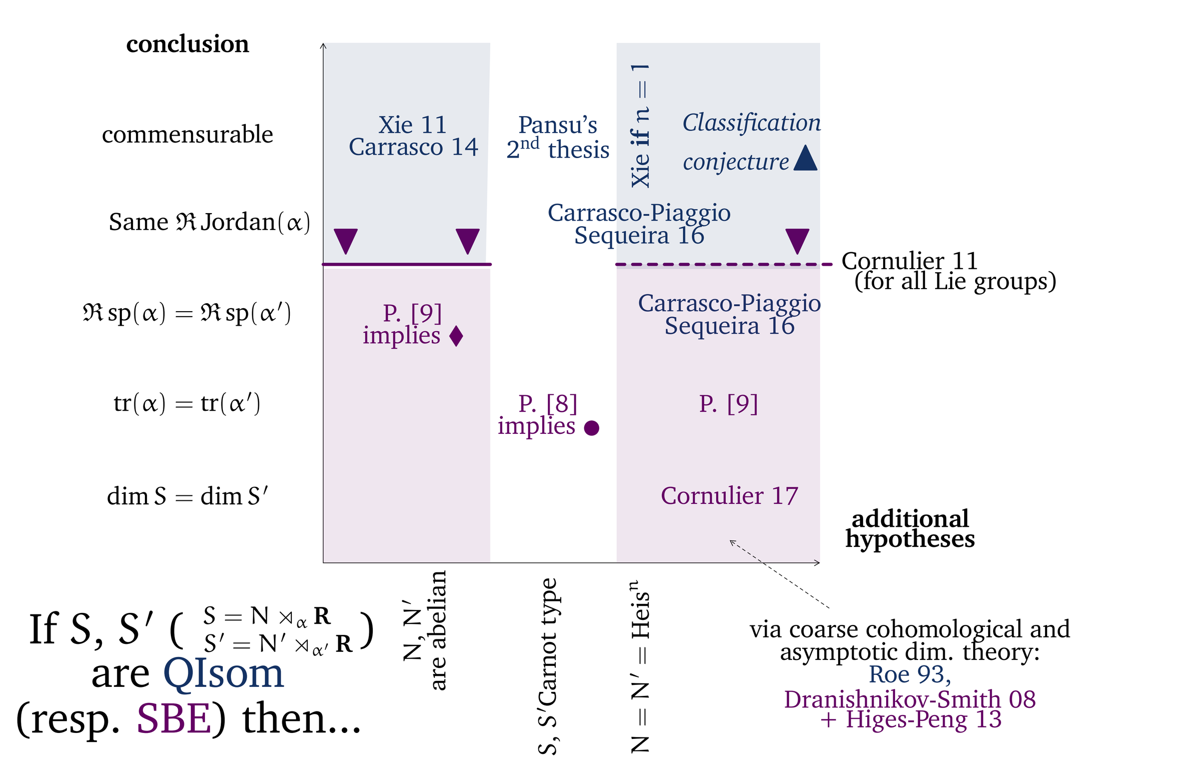


Figure c: QI and SBE classifications of the negatively curved homogeneous spaces (incomplete overview).

2.c. Results

Using **2.a.** we prove (see Figure c):

- ● If $\mathbb{H}^n_{\mathbf{K}}$ and $\mathbb{H}^m_{\mathbf{L}}$ are SBE, then they are homothetic.
- ♦ If $S = N \rtimes_\alpha \mathbf{R}$ and $S' = N \rtimes_{\alpha'} \mathbf{R}$ are SBE and N, N' are abelian then S_∞ and S'_∞ are isomorphic, where S_∞ is obtained from S by erasing the nilpotent parts in the real part Jordan form of α.

● and ♦ answer to questions by Druţu, resp. Cornulier.

3. Back to nilpotent groups

Joint work in progress with C. Llosa Isenrich and R. Tessera. The statement ★ and Pansu's 2nd thesis classify nilpotent Lie groups up to SBE but a quantitative problem remains: can one estimate the best (i.e. infimal) exponent e? Here is an example.

Let G be the simply connected nilpotent Lie group with Lie algebra presentation

$$\langle x_1, x_2, x_3, z, y_1, y_3 \mid [x_1, x_2] = x_3, [x_1, x_3] = [y_1, y_3] = z \rangle.$$

G has class 3; its associated Carnot group G_∞ has a 4-nilpotent central extension implying that it has a Dehn function $\delta_{G_\infty}(n) \asymp n^4$; in contrast G has no such extension and in fact we have $\delta_G(n) \asymp n^3$. A similar low Dehn function phenomenon is known for the higher dimensional Heisenberg groups and other central products [12]. We can deduce that there exists $\varepsilon > 0$ such that there is no $O(r^\varepsilon)$-SBE $G \to G_\infty$. Current work aims at treating more central products of class $\geqslant 3$ and optimizing ε.

4. Some open questions

- Let Γ be a nilpotent or hyperbolic finitely generated group. Is first passage percolation on a Cayley graph of Γ almost surely SBE to Γ? [1].
- Is there no $v(r) \ll \log(r)$ such that $\mathbf{R}^2 \rtimes_A \mathbf{R}$ and $\mathbb{H}^3_{\mathbf{R}}$ are $O(v)$-SBE?
- Does "rigidity of SBE" (in the appropriate sense) occur?

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