Looking at Lie groups through Gromov's telescope

Gabriel Pallier, Karlruher Institut für Technologie

University of Bern, Colloquium

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Microscopes and telescopes

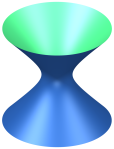
$x^2 + y^2 = 1$

A

$(x^2 + y^2)^2 = z^2 + 1$

B

$x^2 + y^2 = z^2 + 1$



C

Where is the intruder?

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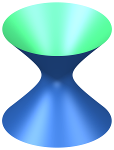
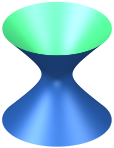


C

Where is the intruder?

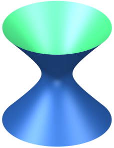
It depends if you are using a microscope or a telescope.

Microscopes and telescopes

	 A	 B	 C
Microscope			
Telescope			

Microscopes and telescopes

We change the rules: now we can move the microscope/telescope

	 A	 B	 C
Microscope			
Moving telescope			

Connected Lie groups

You already know some Lie groups. Let's gather a set of "familiar" Lie groups.

Connected Lie groups

In principle (Levi-Malcev) one could classify all Lie groups if one could classify semi-simple Lie groups and solvable Lie groups.

semi-simple
Nice classification. Not too many.

nilpotent and solvable
No classification. Wild.

Connected Lie groups

semi-simple

nilpotent and solvable

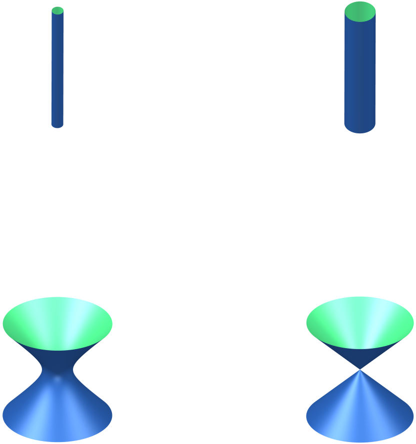
Quasiisometry

Since Mostow, following Margulis and Gromov, a concept has emerged for comparing groups on the large scale: **quasiisometry**.

Quasiisometry

Let X and Y be two metric spaces. $\phi\colon X \to Y$ is a quasiisometry if there exists $L \geqslant 1$ and $C \geqslant 0$ such that

- $\forall x, x' \in X$, $\frac{1}{L} d(x, x') - c \leqslant d(\phi(x), \phi(x')) \leqslant L d(x, x') + c$.
- $\forall y \in Y$, $d(y, \phi(X)) \leqslant c$.



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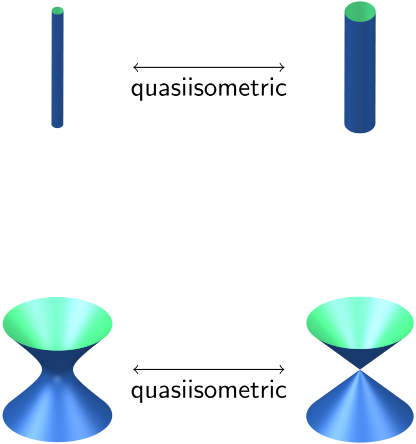

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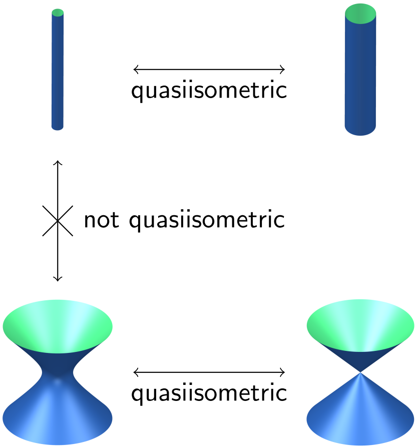


Quasiisometries and the telescope (asymptotic cone)

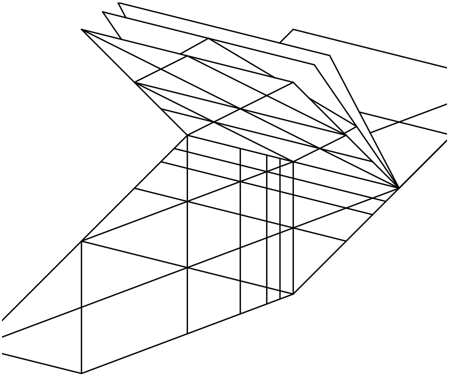
Informally, $\phi\colon X \to Y$ is a quasiisometry if it "goes through any moving telescope".
If you point out at a sequence of points $\{\phi(x_n)\}$ on Y and rescale by a sequence $\{s_n\}$ with limit $+\infty$, you will see a biLipschitz homemorphic image of what you see when pointing the telescope at $\{x_n\}$ on X and rescaling by $\{s_n\}$.

Classifying Lie groups up to quasiisometry

Theorem (Mostow, 1970s)

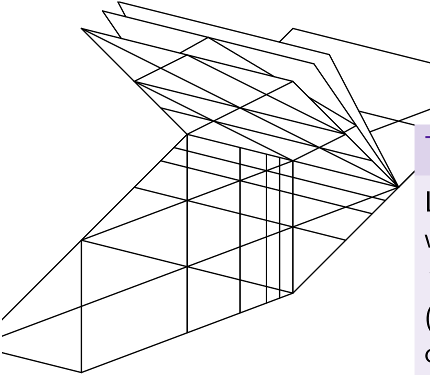
Conjecture (Cornulier)

Looking at semisimple Lie groups through the telescope



Piece of Euclidean building,
$G = \mathrm{SL}(3, \mathsf{R})$,
$r = 2$

Looking at semisimple Lie groups through the telescope



Piece of Euclidean building,
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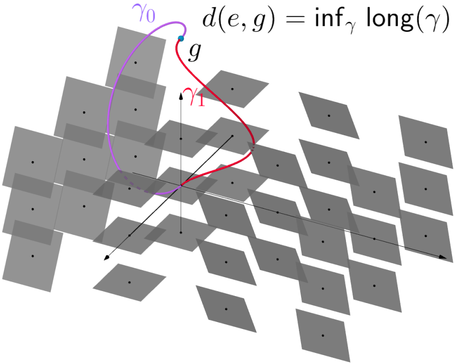
Theorem (Kleiner-Leeb 94)

Let G be a semisimple Lie group with trivial center and rank r.  (G) is a Euclidean building (not locally compact) of covering dimension r.

Looking at nilpotent groups through the telescope



Carnot-Carathéodory metric on the Heisenberg group

Sublinear equivalence (Cornulier,2008)

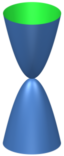
$O(u)$-equivalence

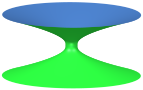
Let X and Y be pointed metric spaces. $\phi\colon X \to Y$ is a $O(u)$-equivalence if $L \geqslant 1$ and u **sublinear function** such that

- $\forall x, x' \in B_X(r)$, $\frac{1}{L} d(x,x') - u(r) \leqslant d(\phi(x), \phi(x')) \leqslant L d(x,x') + u(r)$.
- $\forall y \in B_Y(r)$, $d(y, \phi(X)) \leqslant u(r)$.

Quasiisometry : $u \equiv 1$.

$(x^2 + y^2)^2 = z^2$





catenoid

plane

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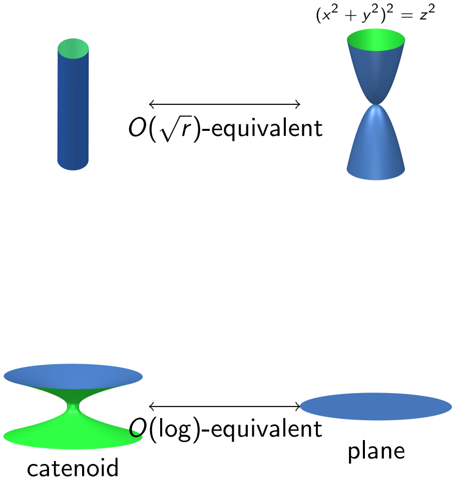

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Sublinear equivalence (Cornulier,2008)

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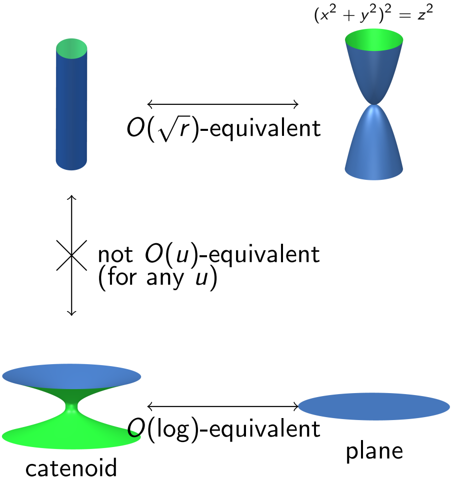

$O(u)$-equivalence and semisimple groups

Theorem (P., 2018)

$O(u)$equivalence and nilpotent groups

Theorem (Pansu, 1989)

$O(u)$equivalence and nilpotent groups

Theorem (Pansu, 1989)

Pansu's microscope

A bilipschitz homeomorphism between Carnot groups is a.e. Pansu differentiable.

$\varphi : G \to H$ is "differentiable" at $\xi \in G$ si

$$D_{\mathrm{P}}\varphi(\xi) : u \mapsto \lim_{t \to +\infty} e^{t\delta_H}\varphi(\xi)^{-1}\varphi(\xi e^{-t\delta_G} u)$$

converges uniformly on every compact set of G.

An (old) application of this

Gromov's polynomial growth theorem (1980)

Let Γ be a finitely generated groups. If Γ has poynomial growth, then it has a nilpotent finite index subgroup.

The proof uses the fact that the image of Γ from infinitely far away is a Lie group (esp. locally compact).

Still open question

Classify finitely generated groups up to quasiisometry.

Another invariant: The filling function

Let $L \geqslant 1$. Let G be a simply connected Lie group with a left-invariant Riemannian metric. Define

$$\mathrm{Fill}_G(L) = \sup_{\gamma:S^1\to G, \ell(\gamma)\leqslant L} \inf\{\mathrm{Area}(\Delta) : \partial\Delta = \mathrm{im}(\gamma)\},$$

If $\mathrm{Fill}_G(L) \sim L^p$ for some $p > 1$ and G and H are quasiisometric, then $\mathrm{Fill}_G(L) \sim L^p$ as well.

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Theorem (LLosa Isenrich - P. -Tessera 2020)

For all $q \in \{3, 4, 5, ...\}$ there exists a pair $\{G, H\}$ of simply connected nilpotent Lie groups with  $(G) \simeq$  (H), $\mathrm{Fill}_G(L) \sim L^q$ but $\mathrm{Fill}_H(L) \sim L^{q+1}$.

Moving away from the groups

"This space [the finitely generated group Γ with its word metric] may appear boring and uneventful to a geometer's eye. To regain the geometric perspective, one has to change his/her position and move the observation point far away from [the group]. Then [...] the points of Γ coalesce into a connected continuous solid unity which occupies the visual horizon without gaps or holes, and fills our geometer's heart with joy."

Misha Gromov

Thanks!