ON THE FINITELY GENERATED GROUPS QUASIISOMETRIC TO BIFILIFORM BY CYCLIC GROUPS

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In his thesis, Ferragut proves structural statement on quasiisometries between (and of) solvable Lie groups and horospherical products, allowing some progress on their quasiisometry classification. The goal of this appendix is to give one further application, this time towards quasiisometric rigidity, given by combining Ferragut's theorem 5.11 and the litterature, namely the main result of Dymarz-Xie's work [DX16], and the work by Xie on quasiconformal maps on filiform groups [Xie15].

In the following Theorem, for $n \geqslant 2$ we let F_n denote the model filiform group of class n, and let δ_n denote a Carnot derivation of its Lie algebra $\mathfrak{f}_n$ (See Section 1 for definitions). Combining Ferragut's theorem with the main theorem in [DX16], we obtain the following.

Theorem 0.1 (After Ferragut and Dymarz-Xie)**.** *Let n, m be positive integers such that $3 \leqslant n < m$. Then, no finitely generated group is quasiisometric to the group $G_{n,m} = (F_n \times F_m) \rtimes_{(\delta_n, -\delta_m)} \mathbb{R}$.*

We call the groups $G_{n,m}$ bifiliform by cyclic. Theorem 0.1 is one case of the first part of the conjectural statement [DPX22, 1.2.2 (2)] and as such, a small step towards quasiisometric rigidity. If F_0 denotes the trivial group, then we can allow $n = 0$ as well (provided $m \geqslant 3$): a stronger statement actually holds [DX16, Theorem 5.8]. The full conclusion of quasiisometric rigidity, namely that a finitely generated group quasiisometric to some $G_{n,m}$ for $3 \leqslant n \leqslant m$ is virtually a lattice in $G_{n,n}$, is to be expected, but seems currently out of reach; we provide a few comments on this in Section 2.

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1. Proof of Theorem 0.1, after Ferragut and Dymarz-Xie

The outline of the proof is the same as in [EFW12, Theorem 7.3] dealing with the group $\mathrm{Sol}(m, n)$. We expand slightly the argument, while replacing Hinkkanen's theorem by [DX16, Theorem 1.1] and Eskin-Fisher-Whyte's description of $\mathrm{QI}(\mathrm{Sol}(m, n))$ with Ferragut's description of $\mathrm{QI}(G_{n,m})$ when $m \neq n$.

Let $n \geqslant 2$. The model filiform group F_n is the simply connected nilpotent Lie group with Lie algebra $\mathfrak{f}_n$. The latter has basis $(e_1, \ldots, e_{n+1})$ where $[e_1, e_j] = e_{j+1}$ for $2 \leqslant j \leqslant n$. Let δ be the Carnot derivation such that $\delta e_1 = e_1, \delta e_2 = e_2$ and

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$\delta e_j = (j-1)e_j$ for $3 \leqslant j \leqslant n+1$. We denote by $\mathrm{Aff}_\delta(F_n)$ the group of maps $F_n \to F_n$ of the form $L_g \circ \exp(t\delta)$ for some $t \in \mathbb{R}$, where L_g denotes the left translations by some $g \in F_n$; $\mathrm{Aff}_\delta(F_n)$ is isomorphic to the Carnot-type Heintze group over F_n. Note that $\mathrm{tr}(\delta) = 1 + \frac{n(n+1)}{2}$.

For $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, let $h_{\epsilon_1,\epsilon_2}$ be the automorphism of $\mathfrak{f}_n$ defined by

$$\begin{cases} h_{\epsilon_1,\epsilon_2}(e_1) = \epsilon_1 e_1, \\ h_{\epsilon_1,\epsilon_2}(e_j) = \epsilon_1^{j-2}\epsilon_2 e_j \quad 2 \leqslant j \leqslant n+1 \end{cases}$$

The $h_{\epsilon_1,\epsilon_2}$ generate a Viergruppe V. We may as well consider this group as a group of automorphisms of F_n. Let us recall the following statement from [DX16, p.1132], which is not hard to check.

Lemma 1.1. *The sequence*

$$1 \longrightarrow \mathrm{Aff}_\delta(F_n) \to \mathrm{Sim}(F_n) \to V \longrightarrow 1.$$

is exact.

Here $\mathrm{Sim}(F_n)$ denotes the group of similarities (or metric homotheties) of the Carnot-Carathéodory metric on F_n for which e_1, e_2 form an orthonormal basis of the horizontal distribution. The lemma implies that this group is almost connected and that $\mathrm{Aff}_\delta(F_n)$ is its unit component.

Let Γ be a finitely generated group quasiisometric to

$$G_{n,m} = (F_n \times F_m) \rtimes \mathbb{R}.$$

Then Γ uniformly quasiacts on any model space of $G_{n,m}$, in particular on the horospherical product metric in which the standard bases in $\mathfrak{f}_n$ and $\mathfrak{f}_m$ are orthonormal, orthonormal to each other, and to the $\mathbb{R}$ direction (once a section has been fixed for the latter). Precisely, let $f \colon \Gamma \to G_{m,n}$ be the quasiisometry, and $\widehat{f}$ its coarse inverse; then for each $\gamma \in \Gamma$ we have that

$$T_\gamma = f \circ L_\gamma \circ \widehat{f}$$

is a quasiisometry of $G_{n,m}$, where $L_\gamma \colon \Gamma \to \Gamma$ is the left multiplication by γ.

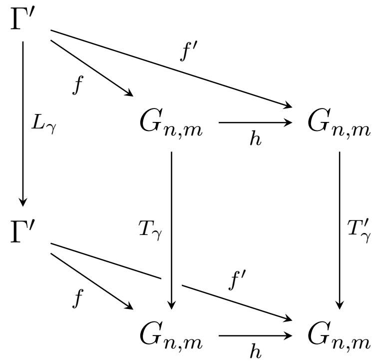
Assume $n \neq m$ and proceed towards a contradiction. By Ferragut's theorem 5.11 applied to the group $G_{n,m}$ and its self-quasiisometries T_γ, we get a homomorphism

$$\rho : \Gamma \to \mathrm{Bilip}(F_n) \times \mathrm{Bilip}(F_m)$$

which to γ associates the boundary maps of T_γ on the upper and lower boundaries. By Dymarz and Xie's theorem [DX16, Theorem 1.1], and Lemma 1.1 recalled above, there is an index 16 subgroup Γ' of Γ and a homomorphism

$$\rho' : \Gamma' \to \mathrm{Aff}_\delta(F_n) \times \mathrm{Aff}_\delta(F_m).$$

A priori, the boundary actions ρ and ρ' are only quasisymmetrically conjugate; but in the case of filiform groups, quasiconformal maps are bilipschitz, as shown by Xie [Xie15]. Extending the pair of bilipschitz conjugating maps to the interior, we get a quasiisometry h of $G_{n,m}$ and the following diagram.



Fix a word distance d on Γ' and let $\lambda \geqslant 1$ and c be such that f' and its coarse inverse $\widehat{f'}$ are both (k,c)-quasiisometries in the sense of Definition 2.1. It follows that the additive quasiisometry constant of T'_γ is at most $kc+c$ for all $\gamma \in \Gamma'$, while the multiplicative constant is at most k^2. The image of ρ' cannot lie anywhere in $\mathrm{Aff}_\delta(F_n) \times \mathrm{Aff}_\delta(F_m)$; ρ' must reach a subgroup of pairs of boundary maps that extend to quasiisometries with a uniform bound on additive quasiisometry constants. We claim that $\rho'(\Gamma')$ is contained in the subgroup

$$L = \{g \in (F_n \rtimes_{\delta_n} \mathbb{R}) \times (F_m \rtimes_{\delta_m} \mathbb{R}) : \exists (x,y) \in F_n \times F_m, \exists t \in \mathbb{R}, g = (x,t,y,-t)\}$$

which is isomorphic to $G_{n,m}$ and whose action by quasiisometries on $G_{n,m}$ is the left multiplication. This is a consequence of the following Lemma.

Lemma 1.2. *Let Q be a group of self-quasiisometries of $G_{n,m}$ with uniformly bounded multiplicative and additive constants. Then there exists a quasi-character $\eta : Q \to \mathbb{R}$ such that*

$$\sup_{x \in G_{n,m}, \Phi \in Q} ||h(\Phi(x)) - h(x)| - \eta(\Phi)| < +\infty.$$

Proof. To every Φ in Q, Ferragut associates two functions that are denoted f_1 and f_2 and defined in the proof of Lemma 5.6. When the quasiisometry constants of Φ are bounded, the difference between f_1 and f_2 is bounded, and Ferragut proves that the quasiisometry Φ is at bounded distance, say K, from a quasiisometry which moves the height by some t_0: this is the conclusion of Corollary 5.9, where the quotients of the real parts of the traces of A_1 and A'_1 is 1 in our case. The latter t_0 can be defined as $f_1(0)$ or $f_2(0)$ (which differ by a bounded amount). Now it is a consequence of the bound expressed in Corollary 5.9 that K is bounded by some constant only depending on the quasiisometry constants k and c of Φ. So we can define $\eta(\Phi) := t_0$, where t_0 is as in the statement of Corollary 5.9. □

Consider the map π from $(F_n \rtimes_{\delta_n} \mathbb{R}) \times (F_m \rtimes_{\delta_m} \mathbb{R})$ to $\mathbb{R} \times \mathbb{R}$ defined by $\pi(x,t,y,s) = (t,s)$. Applying Lemma 1.2 to $Q = \{T'_\gamma\}_{\gamma \in \Gamma'}$ we find that the image of $\pi \circ \rho$ must lie at a finite distance from the line $\{(t,-t) : t \in \mathbb{R}\}$. Since $\pi \circ \rho$ is a group homomorphism, the image of $\pi \circ \rho$ must be contained in this line, so that $\rho'(\Gamma')$ is contained in L.

We need to prove that the image of ρ' is a lattice in $G_{n,m}$, and that its kernel is finite. This is done exactly following the lines of [EFW12]: we have to check that the homomorphism ρ' is proper (which implies that the kernel is finite and the image is discrete), and co-compact (which will achieve showing that $\rho'(\Gamma')$ is a lattice). To see that ρ' is proper, observe that it is by definition the f'-conjugate of the left action of Γ' on itself by left-translations, which is proper; and since f is a quasiisometry, the same property follows for ρ'. Similarly, the left translation action of Γ' on itself is co-compact; the same follows for ρ', since f has co-bounded image.

Finally, note that $G_{n,m}$ is unimodular (and hence, can contain a lattice) if and only if $n = m$. Since we proved that $\rho'(\Gamma')$ is such a lattice, we conclude that $n \neq m$ was not possible.

2. Final comments

Theorem 0.1 is a quasiisometric rigidity statement expressed in a negative form. We would like to emphasize that obtaining the traditional form of quasiisometric rigidity for the class of groups $\mathcal{S}$ (completely solvable, and not Gromov-hyperbolic) encompassing $\mathrm{Sol}(m,n)$ and $G_{m,n}$, may be described as a two-step process, whose completion need not be chronological:

(1) First, show that the non-unimodular groups in $\mathcal{S}$ are not quasiisometric to any finitely generated groups. This is what was done in [EFW12], and the present appendix.
(2) Then, show that any finitely generated group quasiisometric to a unimodular group in $\mathcal{S}$, is virtually a lattice there. Note that even the latter do not always have lattices (in general this can be determined using Auslander's criterion [Aus73, III.6]). In the case of Sol this step is significantly harder (compare [EFW13] to [EFW12]).

A similar two-step process occurs, though with a slightly different mechanism, when obtaining quasiisometric rigidity for Gromov-hyperbolic completely solvable groups. There, what is expected can be stated as follows: any finitely generated group quasiisometric to a negatively curved, isometrically homogeneous Riemannian manifold X, should be virtually a uniform lattice in a rank one Lie group, of which X is the associated symmetric space. The analogue of the second step was achieved in the 1980s and early 1990s by contributions of Tukia, Pansu, Gabai, Casson-Jungres and Chow. The first step is not complete as of now; it is usually reached through the pointed sphere conjecture, explicitely stated by Cornulier [Cor18], and has been an active stream of research in the last decade, led by Xie (of which [Xie15] is a sample result), with an important contribution by Carrasco Piaggio [CP17] essentially showing that X as above should belong to the class of Carnot-type homogeneous spaces of negative curvature.

Finally, one may hope for quasiisometric rigidity statements in which the mystery group Γ is assumed locally compact rather than finitely generated. In the negative curvature case, Cornulier proved that the classical work of the authors cited above, plus [KL09], yield the desired description of compactly generated locally compact groups quasiisometric to symmetric spaces [Cor18]. Beyond symmetric spaces, the Dymarz-Xie theorem [DX16, Theorem 5.8] is currently a rare example of a QI rigidity statement allowing this generality.

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