EXERCISES FOR THE AUSSOIS MINICOURSE "NILPOTENT LIE GROUPS AND SUBLINEAR BILIPSCHITZ EQUIVALENCE"

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A table of Lie algebras occurring in the exercises can be found in the end.

1. Nilpotent Lie algebras

Exercise A. Let $\mathfrak{g}$ be a Lie algebra. Using the Jacobi identity, check that $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a homomorphism of Lie algebras.

Exercise B. Show that every 4-dimensional nilpotent Lie algebras is isomorphic either to $\mathbf{R}^4$ (by which we mean the Lie algebra $\mathbf{R}^4$ with the Lie bracket 0), $\mathfrak{heis} \times \mathbf{R}$ or $\mathfrak{l}_4$, and that the latter three Lie algebras are pairwise non-isomorphic.

Exercise C (Higher-dimensional Heisenberg Lie algebras)**.** Let $\mathfrak{h}$ be a 2-nilpotent Lie algebra such that $\dim(Z(\mathfrak{h})) = 1$.

(1) Show that $\dim \mathfrak{h}$ is odd. Let n be such that $\dim \mathfrak{h} = 2n+1$ in the sequel.
(2) Show that $\mathfrak{h}$ has a basis $(X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z)$ such that $Z \in Z(\mathfrak{g})$,
$$[X_i, Y_j] = \begin{cases} Z & i = j \\ 0 & \text{otherwise;} \end{cases}$$
draw the diagram representing $\mathfrak{h}$ in this basis.
(3) Show that $\mathfrak{h}$ is isomorphic to a Lie subalgebra of $\mathfrak{u}_{n+2}$.

2. Carnot Lie algebras

Exercise D. (1) Show that $(\mathfrak{l}_5')_\infty \simeq \mathfrak{l}_5$. (See table in the end.)
(2) Show that $\mathfrak{l}_5$ and $\mathfrak{l}_5'$ are not isomorphic Lie algebras. (Hint: you may look at ranks of ad_X for various X in $\mathfrak{l}_5$ and $\mathfrak{l}_5'$.)

Exercise E. (1) Show that every 2-nilpotent Lie algebra is Carnot.
(2) Let $\mathfrak{g}$ be a s-nilpotent Lie algebra and let
$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s = \mathfrak{g}_1' \oplus \cdots \oplus \mathfrak{g}_s'$$
be two Carnot gradings on $\mathfrak{g}$. Show that there exists a Lie algebra automorphism of $\mathfrak{g}$ sending $\mathfrak{g}_i$ onto $\mathfrak{g}_i'$ for all $i = 1, \ldots, s$.

3. Nilpotent Lie groups

Exercise F. The aim of this exercise is to prove that a Carnot Lie algebra $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ has a faithful embedding in $\mathfrak{u}_n$ without resorting to Ado's theorem.

(1) Let $\mathfrak{h} = \langle T \rangle \oplus \mathfrak{g}$ where $[T, X] = iX$ for all $X \in \mathfrak{g}_i$. Show that $\mathfrak{h}$ is a Lie algebra and that $\{X \in \mathfrak{h} : \mathrm{ad}_X = 0\} = 0$ where $\mathrm{ad}_X(Y)$ denotes $[X, Y]$.

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(2) Show that $\mathrm{ad}: \mathfrak{h} \to \mathfrak{gl}(\mathfrak{h})$, $X \mapsto \mathrm{ad}_X$ defines an injective Lie algebra homomorphism, i.e. $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}$. Deduce from there that $\mathfrak{g}$ is isomorphic to a Lie subalgebra of $\mathfrak{gl}_{\dim \mathfrak{g}+1}(\mathbf{R})$.
(3) Show that there is a basis $(e_0, \ldots, e_{\dim \mathfrak{g}})$ of $\mathfrak{h}$ such that $\mathrm{ad}(\mathfrak{g})$ is contained in $\mathfrak{k} = \mathrm{span}(E_{00}, \{E_{ij} \colon 1 \leqslant i < j \leqslant \dim \mathfrak{g}\})$ (where $E_{ij} = e_i^\star \otimes e_j$), and then that $\mathfrak{k}$ is isomorphic to a Lie subalgebra of $\mathfrak{u}_{\dim \mathfrak{g}+2}$. Conclude.
(4) Adapt the previous argument to show the result when $\mathfrak{g}$ is the Lie algebra $\mathfrak{l}_5'$ from Exercise D. (First, why is it an extra case?) (Hint: check that $\mathfrak{l}_5'$ admits a grading in the positive integers.)

Exercise G. (1) Express the group law in terms of matrix coefficients on the Heisenberg group $\mathbf{Heis}(\mathbf{R}) = \exp(\mathfrak{u}_3) < \mathrm{GL}_3(\mathbf{R})$, and recover the expression of the group law in exponential coordinates. (Hint : remember about the series expansion of log).
(2) Using the same method with another group, explain why the coefficient $\pm\frac{1}{12}$ appears in front of the Lie monomials of length 3 in the BCH series.

4. Homogeneous norms and growth

Exercise H. In this exercise, you may use Losert's theorem from the first lecture.
(1) Compute the order of polynomial gowth of the Lie groups having Lie algebras $\mathfrak{heis}$, $\mathfrak{l}_4$, and $\mathfrak{g}_{4,3}$.
(2) Let $\Gamma = \langle S \rangle$ be a finitely generated group. Assume that $|S^n| = O(n^\pi)$. Show that Γ has a free-abelian finite-index subgroup. (We recall that $\pi < 3.2$.)
(3) Let Γ be a finitely generated group quasiisometric to $\mathbf{Z}^d$, $d \in \{1, 2, 3\}$. Show that Γ has a finite-index subgroup isomorphic to $\mathbf{Z}^d$.

5. Asymptotic cones and SBE

Exercise I. Let X, Y, Z be pointed metric spaces. Let $f \colon X \to Y$ be a $O(r^e)$-SBE and let $g \colon Y \to Z$ be a $O(r^{e'})$-SBE. Show that $g \circ f \colon X \to Z$ is a $O(r^{\max(e,e')})$-SBE.

Exercise J. (*)
(1) Let Γ be a finitely generated group QI to $\Lambda = \mathbf{Heis}(\mathbf{Z}([\sqrt{2}])$. Show that Γ has a finite-index subgroup isomorphic to a lattice in $L = \mathbf{Heis}(\mathbf{R})^2$. Do Γ and Λ have isomorphic finite-index subgroups?
(2) In this question you are allowed to use that having polynomially bounded growth is a SBE invariant of CGLC groups [1, Corollary 3.5]. Let Γ be a finitely generated group $O(r^e)$-SBE to $\mathbf{Z}^d$ for some $e \in [0, 1)$ and $d \in \mathbf{N}$. Show that Γ has a finite-index subgroup isomorphic to $\mathbf{Z}^d$.

Name	Nonzero brackets (omiting antisymmetry)
$\mathfrak{heis} = \mathfrak{u}_3 = \mathfrak{l}_3$	$[X, Y] = Z$
$\mathfrak{l}_n$, $n \geqslant 3$	$[X_1, X_i] = X_{i+1}, 2 \leqslant i \leqslant n-1$
$\mathfrak{l}_5'$	$[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, and $[X_1, X_4] = [X_2, X_3] = X_5$
$\mathfrak{g}_{4,3}$	$[X_1, X_2] = X_3$, $[X_1, X_3] = [Y_1, Y_2] = Z$.

References

[1] Y. Cornulier. On sublinear bilipschitz equivalence of groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 52(5):1201–1242, 2019.

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