

# NILPOTENT LIE GROUPS AND SUBLINEAR BILIPSCHITZ EQUIVALENCE

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ABSTRACT. As it turns out, the quasiisometric classification of compactly generated locally compact groups of polynomial growth reduces to that of simply connected nilpotent Lie groups. The large-scale geometry of these groups exhibits some sublinear features that we will explore in these lectures. In particular, we will see how sublinear bilipschitz equivalences (generalized quasiisometries) arise between certain pairs of simply connected nilpotent Lie groups, and reframe the quasiisometric classification problem into a quantitative perspective.

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This minicourse is about certain sublinear aspects in the large-scale geometry of nilpotent groups. Our ultimate aim is to explain a simple version of [Cor19, Section 6] and the questions it raises. We only assume knowledge of a few terms in geometric group theory and have made an effort to be otherwise self-contained, although we will admit some results. Some contents need more background to be fully appreciated; they are marked with an asterisk (\*). Some parts of the material we present here have been exposed, with a wider scope, in the following two resources:

- Pierre Pansu’s June 2023 minicourse in Pisa on the large-scale geometry of nilpotent groups, which was recorded and is available here : [Lectures 1 and 2](#) (starting at 0:55:21) and [Lecture 3](#) (starting at 2:49:05). The slides are [available on Pansu’s website](#).
- Chapters 8 to 11 in Enrico Le Donne’s textbook [LD25] (with more prerequisites in differential and metric geometry than here, some of it being covered in the earlier chapters). A fairly complete version of this book is available [on the arXiv](#). There are many exercises.

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These are the lecture notes of a minicourse given in March-April 2026 during the ANR PLAGÉ meeting *Coarse geometry of groups* in Aussois.

In addition, the minicourse likely has some overlaps with Yves Cornulier’s course in the 2019 Jyväskylä summer school and with the minicourse by Claudio Llosa Isenrich and Romain Tessera in the “Équivalence orbitale” meeting at Besse-en-Chandesse in 2024.

## 0. MOTIVATION

Let  $G$  be a compactly generated locally compact (thereafter CGLC) group. We say that  $G$  has **polynomial growth** if for some (any) compact generating set  $S$  of  $G$ ,  $\mathbf{vol}(S^n) \leq \text{poly}(n)$ , where  $\mathbf{vol}$  is some (any) Haar measure on  $G$ . Being of polynomial growth is a quasiisometry (QI) invariant of CGLC groups.

**Theorem 0.1** (... Losert<sup>1</sup> [Los20, Theorems 2 and 3]). *Let  $G$  be a CGLC group of polynomial growth. Then*

- (1)  $G$  has a maximal compact subgroup  $W(G)$  (which is normal in  $G$ ).
- (2)  $G/W(G)$  embeds with closed and co-compact image into a group of the form  $N \rtimes Q$ , where  $N$  is a **simply connected nilpotent Lie group**, and  $Q$  is a compact group.

Moreover, given  $G$  as above,  $N$  is the unique simply connected nilpotent Lie group such that (2) holds.

(We will explain what a simply connected nilpotent Lie group is, but you can take it as a black-box now.) Since  $G/W(G)$  and  $N$  both sit as closed co-compact subgroups in  $N \rtimes Q$ , the groups  $G$  and  $N$  are quasiisometric. We call  $N$  the **connected nilshadow** of  $G$ . Here is a long-standing open problem :

**Problem :** Classify CGLC groups of polynomial growth up to quasiisometry.

An early version was stated in [Pan89] where the problem is attributed to Gromov. More recently the following conjecture has been made. (A slightly weaker form of the conjecture was formulated in [LÖ8] and as a question in [FM00].)

**Conjecture 0.2** (Cornulier, [Cor18]).  *$N$  and  $N'$  are simply connected nilpotent Lie groups. If  $N$  and  $N'$  are quasiisometric, then  $N$  and  $N'$  are isomorphic as Lie groups.*

Conjecture 0.2 would settle the QI classification problem in the following way: Theorem 0.1 associates uniquely two simply connected nilpotent Lie groups  $N$  and  $N'$  to CGLC groups  $G$  and  $G'$ ; if  $G$  and  $G'$  are QI, then  $N$  and  $N'$  are QI as well. According to the conjecture, they should be isomorphic. (Conversely, if  $N \simeq N'$  then  $G$  and  $G'$  are QI). Thus, tentatively,  $G$  and  $G'$  should be QI if and only if their connected nilshadows are isomorphic as Lie groups.

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<sup>1</sup>Historical comments: a) if you replace “polynomial growth” by “nilpotent” in the statement of Theorem 0.1, this is easier and was already known to Guivarc’h in 1973; the general case incorporates Gromov’s well-known theorem on groups with polynomial growth. b) (1) is already [Los01, Prop 1]. An equivalent statement appears in [Cor15, Remark 2.11].

## 1. NILPOTENT LIE ALGEBRAS

Let  $V$  be a finite-dimensional real vector space. A **Lie bracket** on  $V$  is a bilinear map from  $V \times V$  to  $V$ , denoted

$$(X, Y) \rightarrow [X, Y]$$

such that  $[Y, X] = -[X, Y]$  and

$$(J) \quad [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all  $X, Y, Z \in V$ . (Note that antisymmetry implies that  $[X, \lambda X] = [\lambda X, X] = 0$  for all  $\lambda \in \mathbf{R}$ .) A **Lie algebra** is a pair  $\mathfrak{g} = (V, [\cdot, \cdot])$ . A **homomorphism of Lie algebras** is a linear homomorphism which sends the Lie bracket of a pair to the Lie bracket of their images.

**Example 1.1.** For all  $n \geq 1$ ,  $\mathfrak{gl}_n(\mathbf{R}) := (\text{End}(\mathbf{R}^n), (X, Y) \mapsto XY - YX)$  is a Lie algebra. (J) is a consequence of the associativity in  $\text{End}(\mathbf{R}^n)$ .

If  $X$  is an element of a Lie algebra  $\mathfrak{g}$ , we let  $\text{ad}_X = [X, -]$  and call  $\text{ad}_\cdot : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$  the **adjoint representation**. The next exercise explains the terminology.

**Exercise A.** Let  $\mathfrak{g}$  be a Lie algebra. Using (J), check that  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$  is a homomorphism of Lie algebras.

A **Lie monomial** over an alphabet of letters  $\{x_i\}$  is a formal expression of iterated brackets of letters, e.g.  $\mu = [x_1, [x_2, [[x_3, x_4], x_5]]]$  is a Lie monomial over  $\{x_1, \dots, x_5\}$ ; a **simple Lie monomial** is a Lie monomial where the iteration on brackets is always made to the left, e.g.  $\mu$  above is not a simple Lie monomial, while

$$\sigma = [x_1, [x_2, [x_1, [x_3, x_2]]]]$$

is one. To save space we will abbreviate simple Lie monomials and their evaluations by omitting brackets, e.g. we will write  $\sigma = [x_1, x_2, x_1, x_3, x_2]$ . The **length** of a Lie monomial is the number of letters appearing in it, e.g.  $\mu$  and  $\sigma$  have length 5. Using (J) and antisymmetry one may prove that every Lie monomial is formally a linear combination of simple Lie monomials of the same length with coefficients in  $\{\pm 1\}$ . (By “formally” we mean that the situation is exactly analogous to that of polynomial identities holding in any  $\mathbf{R}$ -algebra, or even actually in any ring).

**Definition 1.2** (Lie subalgebra and Lie span). Let  $\mathfrak{g} = (V, [\cdot, \cdot])$  be a Lie algebra. A **Lie subalgebra** of  $\mathfrak{g}$  is a subspace  $W \subset V$  stable under  $[\cdot, \cdot]$ , that we equip with the restriction of the bracket. Given some subset  $S \subseteq \mathfrak{g}$ , we denote by  $\text{Liespan}(S)$  the smallest Lie subalgebra of  $\mathfrak{g}$  containing  $S$  and call it the **Lie subalgebra generated by  $S$** .  $\text{Liespan}(S)$  is the set of formal sums of (simple) Lie monomials with real coefficients, in which we substitute elements of  $S$ .

A Lie subalgebra  $\mathfrak{w}$  of a Lie algebra  $\mathfrak{g}$  is an **ideal** if

$$[\mathfrak{g}, \mathfrak{w}] \subseteq \mathfrak{w}$$

where  $[\mathfrak{g}, \mathfrak{w}]$  denotes  $\text{span}\{[X, Y] : X \in \mathfrak{g}, Y \in \mathfrak{w}\}$ . In this case  $\mathfrak{g}/\mathfrak{w}$  is endowed with the structure of a Lie algebra, by letting  $[X + \mathfrak{w}, Y + \mathfrak{w}] := [X, Y] + \mathfrak{w}$ . Ideals are exactly the kernels of Lie algebras homomorphisms. An example of ideal is the **center**  $Z(\mathfrak{g}) = \ker \text{ad}$ . The following general construction yields a series of nested ideals.

**Definition 1.3.** Let  $\mathfrak{g}$  be a Lie algebra. Define its **descending central series**  $(C^k \mathfrak{g})_{i \geq 1}$  as follows.  $C^1 \mathfrak{g} = \mathfrak{g}$  and, for  $k \geq 1$ ,  $C^{k+1} \mathfrak{g} = [\mathfrak{g}, C^k \mathfrak{g}]$ .

**Lemma 1.4.** For all  $k, \ell \geq 1$ ,  $[C^k \mathfrak{g}, C^\ell \mathfrak{g}] \subseteq C^{k+\ell} \mathfrak{g}$ .

*Proof.* By induction on  $\ell$ . For  $\ell = 1$ ,  $[C^k \mathfrak{g}, C^\ell \mathfrak{g}] = [\mathfrak{g}, C^k \mathfrak{g}] = C^{k+1} \mathfrak{g} \subseteq C^{k+1} \mathfrak{g}$ . Assume it's true for some  $\ell \geq 1$ . Then

$$\begin{aligned} [C^k \mathfrak{g}, C^{\ell+1} \mathfrak{g}] &= [C^k \mathfrak{g}, \mathfrak{g}, C^\ell \mathfrak{g}] \\ &\stackrel{(J)}{\subseteq} \mathfrak{g}, [C^\ell \mathfrak{g}, C^k \mathfrak{g}] + [C^\ell \mathfrak{g}, C^k \mathfrak{g}, \mathfrak{g}] \\ &\subseteq [\mathfrak{g}, C^{k+\ell} \mathfrak{g}] + C^{k+1+\ell} \mathfrak{g} \quad (\text{by the induction hypothesis}) \\ &= C^{k+\ell+1} \mathfrak{g}. \end{aligned} \quad \square$$

**Definition 1.5.** We say that  $\mathfrak{g}$  is **nilpotent** if there exists  $s \in \mathbf{N}$  such that  $C^{s+1} \mathfrak{g} = 0$ . The minimal such  $s$  is called the **nilpotency class** of  $\mathfrak{g}$ ; we also say that  $\mathfrak{g}$  is  $s$ -nilpotent.

For a Lie algebra, being nilpotent passes to subalgebras and quotients. An ordered basis  $(e_1, \dots, e_d)$  of a nilpotent Lie algebra  $\mathfrak{g}$  is **adapted** if for some sequence  $(i_k)$  we have that for all  $k \geq 1$ ,  $C^k \mathfrak{g} = \text{span}(e_{i_k}, \dots, e_d)$ . Every nilpotent Lie algebra has an adapted basis.

**Example 1.6.** Let  $n \geq 2$  and define the following Lie subalgebra of  $\mathfrak{gl}_n(\mathbf{R})$ :

$$\mathfrak{u}_n = \text{span}\{E_{ij} : j > i\} \subseteq \mathfrak{gl}_n(\mathbf{R}).$$

Let us check that  $\mathfrak{u}_n$  is  $(n-1)$ -nilpotent and find an adapted basis. Letting  $E_{i_1 j_1}$  and  $E_{i_2 j_2}$  in  $\mathfrak{u}_n$ , we have

$$[E_{i_1 j_1}, E_{i_2 j_2}] = \begin{cases} E_{i_1 j_2} & i_2 = j_1 \\ -E_{i_2 j_1} & j_2 = i_1 \\ 0 & \text{otherwise,} \end{cases}$$

and if  $i_2 = j_1$  then  $j_2 > i_2 = j_1 > i_1$ , so  $j_2 > i_1$  and  $E_{i_1 j_2} \in \mathfrak{u}_n$ ; the same argument shows that  $E_{i_2 j_1} \in \mathfrak{u}_n$ . This proves that  $\mathfrak{u}_n$  is indeed a Lie algebra. Now note that if  $j_2 \geq i_2 + k$  and  $i_2 = j_1$  then  $j_2 \geq j_1 + k \geq i_1 + k + 1$ ; equality can occur if  $j_1 = i_1 + 1$ , and similarly, This proves that

$$C^k \mathfrak{u}_n = \text{span}\{E_{ij} : j \geq i + k\} = \left\{ \begin{pmatrix} 0 & \cdots & 0 & x_{1,k+1} & \cdots & x_{1,n} \\ \vdots & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & x_{k+1,1} \\ \vdots & & (0) & & & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} : x_{i,j} \in \mathbf{R} \right\}.$$

Especially,  $C^{n-1} \mathfrak{u}_n = \text{span}(E_{1,n}) \neq 0$  and  $C^n \mathfrak{u}_n = 0$ , so  $\mathfrak{u}_n$  is  $(n-1)$ -nilpotent, and an adapted basis is

$$(E_{1,2}, E_{2,3}, \dots, E_{n-1,n}, E_{1,3}, \dots, E_{1,n-1}, E_{2,n}, E_{1,n}).$$

**Proposition and Definition 1.7.** Let  $\mathfrak{g}$  be a  $s$ -nilpotent Lie algebra with  $s \geq 2$ . Then  $\dim \mathfrak{g} \geq s + 1$ . Equality can occur; in this case we call  $\mathfrak{g}$  **filiform**.

*Proof.* Since  $C^{s+1} \mathfrak{g} = 0$  we have

$$(1.1) \quad \dim \mathfrak{g} = \sum_{k=1}^s \dim(C^k \mathfrak{g} / C^{k+1} \mathfrak{g}).$$

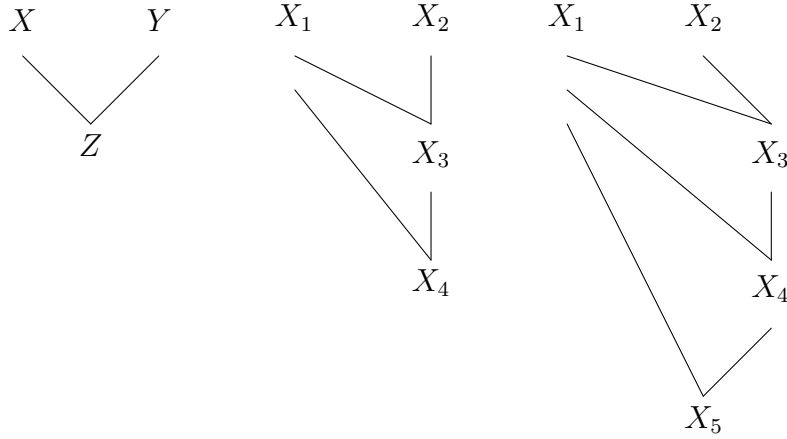


FIGURE 1. Depiction of the defining brackets of model filiform Lie algebras  $\mathfrak{heis} = \mathfrak{l}_3, \mathfrak{l}_4$  and  $\mathfrak{l}_5$  in adapted bases.

Assume towards contradiction that there exists  $k \in \{1, \dots, s\}$  such that  $C^k \mathfrak{g} = C^{k+1} \mathfrak{g}$ , then  $C^\ell \mathfrak{g} = C^k \mathfrak{g}$  for all  $\ell > k$ . Since  $\mathfrak{g}$  is nilpotent we have  $C^k \mathfrak{g} = 0$ . But this contradicts the minimality of  $s$ . So every term in the sum (1.1) is at least equal to 1, and  $\dim \mathfrak{g} \geq s$ .

We now prove that  $\dim(\mathfrak{g}/C^2 \mathfrak{g}) > 1$ . We know that  $s \geq 2$  so there exists some  $Z \in C^2 \mathfrak{g} \setminus C^3 \mathfrak{g}$ . We can write

$$(1.2) \quad Z = [X_1, Y_1] + \dots + [X_r, Y_r]$$

We want to show that there exists  $i \in \{1, \dots, r\}$  such that  $X_i + C^2 \mathfrak{g}$  and  $Y_i + C^2 \mathfrak{g}$  are linearly independent. Otherwise, assuming as we may that the  $X_i$ s are nonzero,  $Y_i \in \lambda_i X_i + C^2 \mathfrak{g}$  for some  $\lambda_i \in \mathbf{R}$ . But then  $[X_i, Y_i] \in [X_i, \lambda_i X_i] + [\mathfrak{g}, C^2 \mathfrak{g}] \subseteq C^3 \mathfrak{g}$ . This is a contradiction, as  $Z \notin C^3 \mathfrak{g}$ . We now show that the equality  $\dim \mathfrak{g} = s + 1$  can occur. Equip  $V = \mathbf{R}^{s+1} = \text{span}(X_1, \dots, X_{s+1})$  with the Lie bracket  $[X_1, X_i] = X_{i+1}$  for  $i = 2, \dots, s$ , and  $[X_k, X_j] = 0$  otherwise. We check (J). If  $\min(k, j) > 1$  then

$$[X_1, [X_k, X_j]] + [X_k, [X_j, X_1]] + [X_j, [X_1, X_k]] = [X_k, X_{j+1}] + [X_j, X_{k+1}] = 0$$

(with the convention  $X_{s+2} = 0$ ). And for every  $j > 1$

$$[X_1, [X_1, X_j]] + [X_1, [X_j, X_1]] + [X_j, [X_1, X_1]] = X_{j+2} - X_{j+2} = 0$$

(again with the convention  $X_{s+2} = X_{s+3} = 0$ ). If  $j, k, \ell$  are all greater than 1, then any bracket of  $X_j, X_k$  and  $X_\ell$  in any order is zero. So (J) holds indeed.  $\square$

**Definition 1.8.** We call the Lie algebra we constructed in the proof  $\mathfrak{l}_{s+1}$  and call it the **model filiform** Lie algebra of class  $s$ .

The Lie algebra  $\mathfrak{u}_3$  is isomorphic to  $\mathfrak{l}_3$ . We call it the **Heisenberg** Lie algebra and denote it by  $\mathfrak{heis}$ . This is the prototype of a non-abelian nilpotent Lie algebra.

**Exercise B.** Show that the 4-dimensional nilpotent Lie algebras are isomorphic either to  $\mathbf{R}^4$  (by which we mean the Lie algebra  $\mathbf{R}^4$  with the Lie bracket 0),  $\mathfrak{heis} \times \mathbf{R}$  or  $\mathfrak{l}_4$ , and that the latter three Lie algebras are pairwise non-isomorphic.

There is a convenient way to denote the Lie brackets in a basis of a Lie algebra. See Figure 1. In theory things can get more complicated, but these diagrams are sufficient to denote all nilpotent Lie algebras of dimension less or equal 6.

**Exercise C.** Let  $\mathfrak{h}$  be a 2-nilpotent Lie algebra such that  $\dim(Z(\mathfrak{h})) = 1$ .

- (1) Show that  $\dim \mathfrak{h}$  is odd; we will let  $n$  be such that  $\dim \mathfrak{h} = 2n + 1$ .
- (2) Show that  $\mathfrak{h}$  has a basis  $(X_1, \dots, X_n, Y_1, \dots, Y_n, Z)$  such that  $Z \in Z(\mathfrak{g})$ ,

$$[X_i, Y_j] = \begin{cases} Z & i = j \\ 0 & \text{otherwise;} \end{cases}$$

draw the diagram representing  $\mathfrak{h}$  in this basis.

- (3) Show that  $\mathfrak{h}$  is isomorphic to a Lie subalgebra of  $\mathfrak{u}_{n+1}$ . Is it isomorphic to a Lie subalgebra of  $\mathfrak{u}_n$ ?

We call these Lie algebras the (higher-dimensional) Heisenberg Lie algebras.

## 2. CARNOT LIE ALGEBRAS

**Proposition and Definition 2.1** (Carnot algebra). *Let  $\mathfrak{g}$  be a  $s$ -nilpotent Lie algebra. We define a new Lie algebra,  $\mathfrak{g}_\infty$ , that we will call the **associated Carnot algebra** of  $\mathfrak{g}$ , in the following way. As a vector space,*

$$\mathfrak{g}_\infty = \bigoplus_{k=1}^s C^k \mathfrak{g} / C^{k+1} \mathfrak{g}.$$

For  $X \in C^k \mathfrak{g}$  and  $Y \in C^\ell \mathfrak{g}$ , we define  $\{X + C^{k+1} \mathfrak{g}, Y + C^{\ell+1} \mathfrak{g}\} := [X, Y] + C^{k+\ell+1} \mathfrak{g}$ . After extending by linearity,  $(X, Y) \mapsto \{X, Y\}$  defines a Lie bracket on  $\mathfrak{g}_\infty$ , with the following properties :

**C1.**  $\mathfrak{g}_\infty$  is  $s$ -nilpotent.

**C2.** Denote  $(\mathfrak{g}_\infty)_k := C^k \mathfrak{g} / C^{k+1} \mathfrak{g}$ . Then  $((\mathfrak{g}_\infty)_k)_k$  defines a **grading of  $\mathfrak{g}_\infty$  by the positive integers**:  $\mathfrak{g}_\infty = (\mathfrak{g}_\infty)_1 \oplus \dots \oplus (\mathfrak{g}_\infty)_s$  and  $[(\mathfrak{g}_\infty)_k, (\mathfrak{g}_\infty)_\ell] \subseteq (\mathfrak{g}_\infty)_{k+\ell}$ .

**C3.**  $\mathfrak{g}_\infty = \text{Liespan}((\mathfrak{g}_\infty)_1)$ .

If  $\mathfrak{g} \simeq \mathfrak{g}_\infty$  then we say that  $\mathfrak{g}$  is itself a **Carnot Lie algebra**. A direct sum decomposition of a Lie algebra is called a **Carnot grading** if it has the properties **C2** and **C3** above; a Lie algebra is Carnot if and only if it possesses a Carnot grading.

*Sketch of proof.* We prove that the Lie bracket is well defined in  $\mathfrak{g}_\infty$ . Given  $X \in C^k \mathfrak{g}$  and  $Y \in C^\ell \mathfrak{g}$ , let  $X' \in X + C^{k+1} \mathfrak{g}$  and  $Y' \in Y + C^{\ell+1} \mathfrak{g}$ . Then

$$[X', Y'] - [X, Y] \in [C^k \mathfrak{g}, C^{\ell+1} \mathfrak{g}] + [C^{k+1} \mathfrak{g}, C^\ell \mathfrak{g}] + [C^{k+1} \mathfrak{g}, C^{\ell+1} \mathfrak{g}] \subseteq C^{k+\ell+1} \mathfrak{g}$$

by Lemma 1.4. This shows that the Lie bracket is well defined in  $\mathfrak{g}_\infty$ ; linearity, antisymmetry, and **(J)** follow from that of the Lie bracket in  $\mathfrak{g}$ . Property **C2** follows again from Lemma 1.4. By induction on  $k \geq 1$ , we prove that

$$C^k \mathfrak{g}_\infty = \bigoplus_{i \geq k} (\mathfrak{g}_\infty)_i$$

and that  $[(\mathfrak{g}_\infty)_1, (\mathfrak{g}_\infty)_k] = (\mathfrak{g}_\infty)_{k+1}$ . □

The terminology of Carnot Lie algebra is the most used in geometric group theory; it was introduced by Pansu in a paper concerned with QI rigidity [Pan89]. However, authors from the PDE community (and sometimes also in subriemannian geometry) have studied these Lie algebras since longer and rather call them **stratified Lie algebras**. In relation to certain problems of prolongations of geometric structures, some grade them with *negative* integers [Tan70]. The process of going from  $\mathfrak{g}$  to  $\mathfrak{g}_\infty$  is an example of what physicists have called a **Lie algebra contraction** and it plays a

role when taking certain limits (either from relativistic to classical, or from quantum to classical physics) [IW53, MR206]. One slogan is that the contracted Lie algebra is “more abelian” than the one we started with; for instance, the center increases. We work out a few examples below before discussing a few subtleties.

**Example 2.2.** The Heisenberg Lie algebra, and more generally the nilpotent Lie algebras  $\mathfrak{u}_n$  and  $\mathfrak{l}_n$  are Carnot. This can be checked by making the corresponding gradings explicit; in fact we already have adapted bases (in the case of  $\mathfrak{u}_n$  this was done in Example 1.6), the gradings can be obtained by concatenating linear spans of subsequences of elements of from these bases.

**Example 2.3.** Consider the Lie algebra

$$\mathfrak{g}_{4,3} = \text{span}(X_1, X_2, X_3, Y_1, Y_2, Z)$$

where  $[X_1, X_2] = X_3$ ,  $[X_1, X_3] = [Y_1, Y_2] = Z$ , and all other brackets are zero. Some computations give that  $C^2\mathfrak{g}_{4,3} = \text{span}(X_3, Z)$ , that  $C^3\mathfrak{g}_{4,3} = \text{span}(Z)$ , and that  $\mathfrak{g}_{4,3}$  is 3-nilpotent. Moreover, the basis  $(X_1, X_2, Y_1, Y_2, X_3, Z)$  is adapted, and

$$\{Y_1 + C^2\mathfrak{g}_{4,3}, Y_2 + C^2\mathfrak{g}_{4,3}\} = Z + C^3\mathfrak{g}_{4,3} = 0$$

in  $C^2\mathfrak{g}_{4,3}/C^3\mathfrak{g}_{4,3}$ . It follows that  $(\mathfrak{g}_{4,3})_\infty \simeq \mathfrak{l}_4 \times \mathbf{R}^2$  where the  $\mathfrak{l}_4$  factor is generated by  $X_1, X_2 \bmod C^2\mathfrak{g}_{4,3}$ ,  $X_3 \bmod C^3\mathfrak{g}_{4,3}$  and  $Z$ . A little of linear algebra shows that  $Z(\mathfrak{g}_{4,3}) = \text{span}(Z)$  is 1-dimensional, while  $Z((\mathfrak{g}_{4,3})_\infty)$  is 3-dimensional. So  $\mathfrak{g}_{4,3}$  is not Carnot.

**Exercise D.** Consider the Lie algebra

$$\mathfrak{l}'_5 = \text{span}(X_1, X_2, X_3, X_4, X_5)$$

where  $[X_1, X_2] = X_3$ ,  $[X_1, X_3] = X_4$  and  $[X_1, X_4] = [X_2, X_3] = X_5$ .

- (1) Compute the lower central series of  $\mathfrak{l}'_5$  and show that  $(\mathfrak{l}'_5)_\infty \simeq \mathfrak{l}_5$ .
- (2) Show that  $\mathfrak{l}_5$  and  $\mathfrak{l}'_5$  are not isomorphic Lie algebras.

**Exercise E.** (1) Show that every 2-nilpotent Lie algebra is Carnot.  
 (2) Let  $\mathfrak{g}$  be a  $s$ -nilpotent Lie algebra and let

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s = \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_s$$

be two Carnot gradings on  $\mathfrak{g}$ . Show that there exists a Lie algebra automorphism of  $\mathfrak{g}$  sending  $\mathfrak{g}_i$  onto  $\mathfrak{g}'_i$  for all  $i = 1, \dots, s$ .

**Proposition and Definition 2.4.** Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$  be a Carnot Lie algebra. Decompose every  $X \in \mathfrak{g}$  into  $X_1 + \cdots + X_s$  where  $X_i \in \mathfrak{g}_i$  for all  $i$ . For all  $t \in \mathbf{R}_{>0}$ , define

$$t \bullet X := tX_1 + \cdots + t^s X_s.$$

Then  $t \bullet -$  is a Lie algebra automorphism of  $\mathfrak{g}$ , called a **Carnot dilation**.

*Proof.* Let  $X \in \mathfrak{g}_k$  and  $Y \in \mathfrak{g}_\ell$ . Then  $[X, Y] \in \mathfrak{g}_{k+\ell}$ , so

$$t \bullet [X, Y] = t^{k+\ell}[X, Y] = [t^k X, t^\ell Y] = [t \bullet X, t \bullet Y].$$

We may now conclude that the same identity holds for every  $X, Y \in \mathfrak{g}$  by linearity of the bracket.  $\square$

It is worth noting, at this point, that in  $\mathbf{R}^d$ , the fact that Lie algebra automorphisms (which are just elements of  $\mathfrak{gl}_d(\mathbf{R})$ ) commute with the dilation (which is just the multiplication by scalar  $t$ ) is a consequence of their mere linearity, but in a non-abelian Carnot Lie algebra, it's not. We say that a homomorphism  $\psi: \mathfrak{g} \rightarrow \mathfrak{h}$  between Carnot Lie algebras is a **Carnot homomorphism** if it intertwines the Carnot dilations of  $\mathfrak{g}$  and  $\mathfrak{h}$ , namely, for all  $X \in \mathfrak{g}$  and  $t \in \mathbf{R}_{>0}$ ,  $t \bullet \psi(X) = \psi(t \bullet X)$ .

Although  $\mathfrak{g}$  and  $\mathfrak{g}_\infty$  are isomorphic as vector spaces, there is no canonical isomorphism  $L: \mathfrak{g} \rightarrow \mathfrak{g}_\infty$ , and indeed no canonical isomorphism respecting the lower central series.

### 3. NILPOTENT LIE GROUPS

**Theorem 3.1** (Consequence of Ado's theorems). *Let  $\mathfrak{g}$  be a nilpotent Lie algebra. Then  $\mathfrak{g}$  is isomorphic to a Lie subalgebra of  $\mathfrak{u}_n$  for large enough  $n$ .*

Ado's theorem is quite deep, see e.g. Bourbaki [Bou82, Chapter 1, §7] for a proof.

**Exercise F.** The aim of this exercise is to prove Theorem 3.1 without resorting to Ado's theorem in the special case when  $\mathfrak{g}$  is Carnot.

- (1) Let  $\mathfrak{h} = \mathfrak{g} \oplus \langle T \rangle$  where  $[T, X] = iX$  for all  $X \in \mathfrak{g}$ . Show that  $\mathfrak{h}$  is a Lie algebra and that  $\{X \in \mathfrak{h} : \text{ad}_X = 0\} = 0$  where  $\text{ad}_X(Y)$  denotes  $[X, Y]$ .
- (2) Show that  $\text{ad}: \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{h})$ ,  $X \mapsto \text{ad}_X$  defines an injective Lie algebra homomorphism, i.e.  $[\text{ad}_X, \text{ad}_Y] = \text{ad}_{[X, Y]}$ . Deduce from there that  $\mathfrak{g}$  is isomorphic to a Lie subalgebra of  $\mathfrak{gl}_{\dim \mathfrak{g} + 1}(\mathbf{R})$ .
- (3) Show that there is a basis  $(e_0, \dots, e_{\dim \mathfrak{g}})$  of  $\mathfrak{h}$  such that  $\text{ad}(\mathfrak{g})$  is contained in  $\mathfrak{k} = \text{span}(E_{00}, \{E_{ij} : 1 \leq i < j \leq \dim \mathfrak{g}\})$  (where  $E_{ij} = e_i^* \otimes e_j$ ), and then that  $\mathfrak{k}$  is isomorphic to a Lie subalgebra of  $\mathfrak{u}_{\dim \mathfrak{g} + 2}$ . Conclude.
- (4) Adapt the previous argument to show the result when  $\mathfrak{g}$  is the Lie algebra  $\mathfrak{l}'_5$  from Exercise D. (First, why is it an extra case?) (Hint: check that  $\mathfrak{l}'_5$  admits a grading in the positive integers).

**Proposition and Definition 3.2.** *Let  $\mathfrak{g}$  be a nilpotent Lie algebra. Choose  $\mathfrak{u}$  a Lie subalgebra of  $\mathfrak{u}_n$  isomorphic to  $\mathfrak{g}$ . Then the set*

$$G = \exp(\mathfrak{u}) = \{\exp(X) : X \in \mathfrak{u}\}$$

*is a simply connected, closed subgroup of  $\text{GL}_n(\mathbf{R})$ .  $\exp$  is a homeomorphism onto its image. We define  $G$  to be the (simply connected) **Lie group** associated to  $\mathfrak{g}$ . This does not depend on  $\mathfrak{u}$  in the following way: if  $\mathfrak{u} \subseteq \mathfrak{u}_n$  and  $\mathfrak{u}' \subseteq \mathfrak{u}_m$  as above, and  $G = \exp(\mathfrak{u})$  and  $G' = \exp(\mathfrak{u}')$ , then there is a group isomorphism  $\psi: G \rightarrow G'$  which is the restriction of a polynomial map  $\text{GL}_n(\mathbf{R}) \rightarrow \text{GL}_m(\mathbf{R})$  with polynomial inverse, and commutes with the exponential maps. We also say that  $\mathfrak{g}$  is the **Lie algebra of  $G$** .*

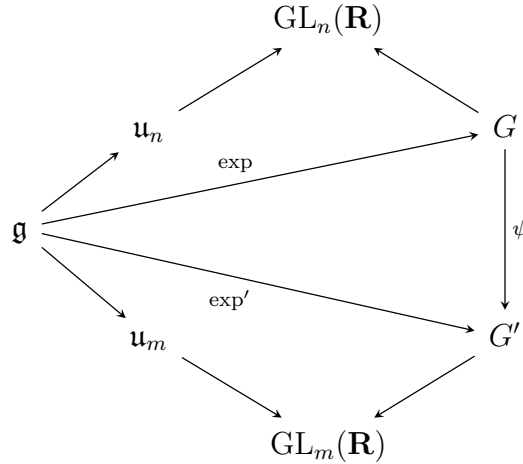
From now on, we adopt the following convention: the same letter denotes a simply connected nilpotent Lie group and its Lie algebra, arabic uppercase and gothic lowercase respectively. We declare that two simply connected nilpotent Lie groups  $G$  and  $H$  are **isomorphic as<sup>2</sup> Lie groups** if and only if  $\mathfrak{g}$  and  $\mathfrak{h}$  are isomorphic Lie algebras.

<sup>2</sup>One can show that if the underlying topological groups of  $G$  and  $H$  are isomorphic as topological groups, then  $G$  and  $H$  are isomorphic as Lie groups. Beware that under Zorn's lemma, the abstract groups  $\mathbf{R}$  and  $\mathbf{R}^2$  are isomorphic and even isomorphic as  $\mathbf{Q}$ -vector spaces.

*Remark 3.3 (\*)*. We will not prove Proposition-Definition 3.2 here; in a more systematic exposition of the theory, this should actually come after the BCH formula that we will soon state. Simply connected nilpotent Lie groups are real points of unipotent algebraic groups (See e.g. [Mil17, Chapter 14]) and there is a well-defined notion of polynomial map (with real coefficients) on a simply connected nilpotent Lie group which does not depend on the embedding in a linear group.

As a consequence of Proposition-Definition 3.2,  $\exp : \mathfrak{g} \rightarrow G$  is well-defined independently of  $\mathfrak{u}_n$  (See diagram on the right:  $\psi$  is the polynomial isomorphism with a polynomial inverse). And thus we will mostly ignore  $\mathfrak{u}_n$  in the sequel. We denote the inverse of  $\exp$  by  $\log : G \rightarrow \mathfrak{g}$  and call it the **logarithm**. If  $G$  is embedded in  $\exp(\mathfrak{u}_n)$  this is the restriction of the usual logarithm

$$\log x = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^n}{n-1} x^{n-1}.$$



In order to make sense of the statement of the next theorem, it is important to note that a formal series of Lie monomials over a finite alphabet has only finitely many terms of a given length, and so there is no convergence issue when substituting its indeterminates by elements of a nilpotent Lie algebra.

**Theorem 3.4** (Baker-Campbell-Hausdorff series). *There exists a formal series of simple Lie monomials over the 2-letter alphabet, denoted H below, with coefficients in Q, such that for any Lie subalgebra of  $\mathfrak{u}_n$ , it holds*

$$\log(\exp X \exp Y) = H(X, Y).$$

Moreover, the series  $H(X, Y)$  cut to the terms of length at most four is

$$(3.1) \quad X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, X, Y] + [Y, Y, X]) - \frac{1}{24}[Y, X, X, Y].$$

See [God04, §6.6] for a proof which does not let the calculation of (3.1) as an exercise to the reader. There is a general version of the formula in non-nilpotent Lie algebras which deals with the convergence issues, see e.g. Bourbaki [Bou82, Chapter 2].

A very important consequence of the Baker-Campbell-Hausdorff series is that it allows us to write the multiplication law of a nilpotent group in **exponential coordinates** on its Lie algebra itself. Let us work out the important case of the Heisenberg Lie algebra  $\mathfrak{heis} = \mathfrak{u}_3 = \mathfrak{l}_3$ . Recall that it is generated with  $E_{12}$ ,  $E_{23}$  and  $E_{13}$  and that  $[E_{12}, E_{23}] = E_{13}$ , the brackets between other pairs of basis elements being zero. Writing  $(p, q, r)$  for the coordinates in this basis, we have

$$\log(\exp(p_1, q_1, r_1) \exp(p_2, q_2, r_2)) = \left( p_1 + q_1, p_2 + q_2, r_1 + r_2 + \frac{p_1 q_2 - p_2 q_1}{2} \right).$$

**Exercise G.** (1) Express the group law in terms of matrix coefficients on the Heisenberg group  $\mathbf{Heis}(\mathbf{R}) = \exp(\mathfrak{u}_3) < GL_3(\mathbf{R})$ , and recover the expression of the group law in exponential coordinates. (Hint : remember about the series expansion of  $\log$ ).

- (2) Using the same method with another group, explain why the coefficient  $\pm \frac{1}{12}$  appears in front of the Lie monomials of length 3 in the BCH series.

From now on, given a nilpotent Lie algebra  $\mathfrak{g}$  and  $x, y \in \mathfrak{g}$ , we will denote

$$x * y := \log(\exp(x) \exp y).$$

We use lowercase for  $x$  and  $y$  to emphasize that we think of them as group elements, even if they are elements of the Lie algebra.

**Proposition 3.5.** *Let  $\mathfrak{g}$  be a nilpotent Lie algebra. Equip it with a Lebesgue measure  $\lambda$  that is invariant under vector space translation. Then  $\lambda$  is also a Haar measure on the nilpotent group  $(\mathfrak{g}, *)$ .*

*Proof.* We need to show that for every  $x \in \mathfrak{g}$ , the left multiplication  $L_x : y \mapsto x * y$  has its Jacobian determinant identically equal to 1. Equip  $\mathfrak{g}$  with an adapted basis  $(e_1, \dots, e_n)$ . Let  $x = (x_1, \dots, x_s)$  and  $y = (y_1, \dots, y_s)$  in the corresponding coordinates. It follows from the BCH formula that

$$x * y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + f_3(x_1, x_2, y_1, y_2), \dots, x_n + y_n + f_n(x_{\hat{n}}, y_{\hat{n}}))$$

where  $f_i$  are polynomials for  $i = 3, \dots, n$  and  $x_{\hat{n}}$  denotes  $(x_1, \dots, x_{n-1})$ . Thus

$$\frac{\partial(L_x)_j}{\partial y_i}(y) = \begin{cases} 0 & i > j \\ 1 & i = j \\ \frac{\partial f_j}{\partial y_i}(x_1, \dots, x_{j-1}, y_1, \dots, y_{j-1}) & i < j. \end{cases}$$

The Jacobian matrix of the left multiplication by  $x$  is triangular with 1s on the diagonal, hence its determinant is 1.  $\square$

**Definition 3.6.** Given a simply connected nilpotent Lie group  $G$ , we denote by  $G_\infty$  the nilpotent Lie group with Lie algebra  $\mathfrak{g}_\infty$ . We call it the associated Carnot group of  $G$ . If  $G$  and  $G_\infty$  are isomorphic as Lie groups (which occurs if and only if  $\mathfrak{g}$  and  $\mathfrak{g}_\infty$  are isomorphic Lie algebras), we say that  $G$  is a Carnot group.

**Proposition 3.7.** *Let  $G = (\mathfrak{g}, *)$  be a Carnot group, and let  $\{t \bullet -\}$  be the 1-parameter group of dilations of  $\mathfrak{g}$ . For every  $t \in \mathbf{R}_{>0}$ ,  $x \mapsto t \bullet x$  defines a Lie group automorphism of  $G$ . We call it the Carnot dilation of  $G$ .*

*Proof.* Using the BCH series,

$$t \bullet (x * y) = t \bullet H(x, y) = H(t \bullet x, t \bullet y) = (t \bullet x) * (t \bullet y).$$

This proves that  $t \bullet -$  is a Lie group endomorphism. Its inverse is  $t^{-1} \bullet -$ .  $\square$

**Definition 3.8.** A Lie group homomorphism  $\Psi: G \rightarrow H$  between Carnot groups  $G$  and  $H$  is a **Carnot group homomorphism** if  $\Psi(t \bullet x) = t \bullet (\Psi(x))$  for all  $x \in G$  and  $t \in \mathbf{R}_{>0}$ .

It is worth noting, again, that in  $\mathbf{R}^d$ , the fact that automorphisms commute with the dilation is a consequence of their mere linearity, but in general, it's not.

*Remark 3.9* (\*). Although this course is not primarily concerned with finitely generated groups, let us mention at this stage that a theorem of Mal'cev ensures that, as soon as the nilpotent Lie algebra  $\mathfrak{g}$  admits a basis  $(e_1, \dots, e_n)$  with

$$[e_i, e_j] = \sum_k c_{i,j}^k e_k$$

where all the coefficients are rational,  $G$  admits a co-compact lattice, which is a finitely generated torsion-free nilpotent group [Mal51]. Explicit presentations of these lattices can be worked out from the BCH series. For instance, the Heisenberg group has a lattice whose presentation is

$$\langle x, y, z \mid [x, z] = [y, z] = 1, [x, y] = z \rangle.$$

This theorem of Mal'cev has a converse which is an ancestor of Theorem 0.1 and which ensures that every finitely generated torsion-free nilpotent group is a lattice in a simply connected Lie group (We refer to A.6 for a precise statement). This lattices is realized as **Heis**( $\mathbf{Z}$ ). More generally, lattices in nilpotent Lie groups are virtually the integer points of unipotent algebraic groups defined over  $\mathbf{Q}$ ; different  $\mathbf{Q}$ -groups with the same real points will yield non-commensurable lattices in a same simply connected nilpotent Lie group.

#### 4. HOMOGENEOUS NORM AND GROWTH

Let  $\mathfrak{g}$  be a nilpotent Lie algebra. Let  $\mathfrak{g}_\infty$  be its associated Carnot graded Lie algebra. Let  $V$  be the vector space underlying  $\mathfrak{g}_\infty$ . Write  $V_i = (\mathfrak{g}_\infty)_i$ .

Let  $\mathfrak{L} : \mathfrak{g} \rightarrow \mathfrak{g}_\infty$  be a linear isomorphism respecting the lower central series filtrations (any choice of adapted basis on  $\mathfrak{g}$  gives rise to such an isomorphism, by mapping every basis element to its image modulo the first term in the central series where it isn't).

On the underlying vector space  $V$  of  $\mathfrak{g}_\infty$  we have :

- (1) The Carnot bracket  $\{X, Y\}_\infty$
- (2) The original bracket  $[X, Y] = \mathfrak{L}[\mathfrak{L}^{-1}X, \mathfrak{L}^{-1}Y]$
- (3) The group law  $(X, Y) \mapsto X *_\infty Y$  coming from the BCH series evaluated on the graded bracket:

$$x *_\infty x = x + y + \frac{1}{2}\{x, y\} + \frac{1}{12}(\{x, x, y\} + \{y, y, x\}) + \dots$$

- (4) The group law  $(x, y) \mapsto x * y$  coming from the BCH series evaluated on the original bracket:

$$x * x = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, x, y] + [y, y, x]) + \dots$$

Note that  $-x$  is unambiguous: it is equal to  $x^{-1}$  for  $*_\infty$  and  $*$ . (Beware however that in general  $-(x * y) \neq (-x) * y$  and also  $-(x *_\infty y)$  unlike when  $*$  is commutative.)

**Definition 4.1.** Fix a norm on  $V$ . Let  $x \in \mathfrak{g}_\infty$ ; let  $x = \underline{x}_1 + \dots + \underline{x}_s$  where  $\underline{x}_i \in V_i$  (We underline the “components”  $\underline{x}_i$  to emphasize that these are still vectors). Define the **homogeneous norm** of  $x$  as

$$[x] := \sup_i \|\underline{x}_i\|^{1/i}.$$

**Theorem 4.2** (Guivarc'h 1973, [Gui73]). *Let  $d$  be a proper,  $*$ -invariant geodesic distance on  $\mathfrak{g}_\infty$ . Then there exists  $\lambda \geq 1$  and  $c \geq 0$  such that  $\frac{1}{\lambda}[x] - c \leq d(0, X) \leq \lambda[x] + c$ .*

Note that the theorem also applies to any  $*_{\infty}$ -invariant distance. A variant of this is known as the “ball-box theorem” [LD25] because it allows to put balls in between boxes, as we shall see. Let’s first give an application.

**Corollary 4.3** (Bass-Guivarc’h formula, modern version). *Let  $G$  be a CGLC group with polynomial growth and let  $S$  be a compact generating set of  $G$ . Then there are constants  $C_+$  and  $C_-$  such that for  $n$  large enough*

$$C_- n^d \leq \text{vol}(S^n) \leq C_+ n^d.$$

Moreover,

$$(4.1) \quad d = \sum_{i=1}^s i \dim C^i \mathfrak{g} / C^{i+1} \mathfrak{g} = \sum_{i=1}^{\infty} \dim C^i \mathfrak{g}$$

where  $\mathfrak{g}$  is the Lie algebra of the connected nilshadow of  $G$ .

*Proof.* Define a box  $\mathbf{B}(r) = \{x \in V : [x] \leq r\}$  where the homogeneous norm  $[\cdot]$  comes from a  $\ell^{\infty}$  norm in an adapted basis. The Haar/Lebesgue volume of a box is

$$\text{vol}(\mathbf{B}(r)) = \prod_{i=1}^s \text{vol}_{\dim V_i} B_{\ell^{\infty}}(0, r^i) = 2^{\dim V} \prod_{i=1}^s r^{i \dim V_i}.$$

By Theorem 4.2,  $\text{vol}(S^n)$  is of the order of  $\text{vol}(\mathbf{B}(n))$ . □

**Exercise H.** In this exercise, you may use Losert’s theorem from the first lecture.

- (1) Compute the order of polynomial growth of the Lie groups having Lie algebras  $\mathfrak{heis}$ ,  $\mathfrak{l}_4$ , and  $\mathfrak{g}_{4,3}$ .
- (2) Let  $\Gamma = \langle S \rangle$  be a finitely generated group. Assume that  $|S^n| = O(n^{\pi})$ . Show that  $\Gamma$  has a free abelian finite-index subgroup.
- (3) Let  $\Gamma$  be a finitely generated group quasiisometric to  $\mathbf{Z}^d$  for  $d \in \{1, 2, 3\}$ . Show that  $\Gamma$  has a finite-index subgroup isomorphic to  $\mathbf{Z}^d$ .

**Example 4.4.** Let us compute the order of growth of  $L_{s-1}$  (which appeared briefly on the board towards the end of Matt’s first talk on Monday). We have

$$(4.2) \quad d = 2 + 2 + 3 + \cdots + s - 1 = 1 + \frac{s(s-1)}{2}.$$

This is the maximal order of growth of a  $(s+1)$ -dimensional connected nilpotent Lie group.

*Remark 4.5.* Let  $G$  be a simply connected nilpotent Lie group. Then  $G$  and  $G_{\infty}$  have the same order of polynomial growth.

**Theorem 4.6** (Goodman, Cornulier [Goo77, Cor19]). *Let  $\mathfrak{g}$  be a nilpotent Lie algebra. There exists  $e = e(\mathfrak{g}) \in [0, 1)$  such that  $|x * y - x *_\infty y| = O(|x|^e + |y|^e)$ .*

Cornulier’s version includes an upper bound on  $e(\mathfrak{g})$  which is in general better than the bound  $e(\mathfrak{g}) \leq 1 - 1/s$  coming out of Goodman’s proof. We will only prove the theorem when  $s = 3$  which is the first case of interest (when  $s = 2$ ,  $G$  is Carnot and the statement follows with  $e = 0$ ). The case  $s = 4$  is the first where Cornulier’s improvement on Goodman’s become noticeable.

*Proof when  $s = 3$ .* Assume  $s = 3$ . For a vector  $X \in V$ , denote  $X_i$  its projection to  $V_i$  in the direct sum decomposition  $V = V_1 \oplus \cdots \oplus V_s$ . Start noticing that for  $X, Y \in V_1$  we have  $[X, Y] \in C^2\mathfrak{g} = V_2 \oplus V_3$  but  $\{X, Y\} \in V_2$ . This is, in fact, the only difference between the two brackets. Precisely we compute that

$$[X, Y] - \{X, Y\} = [X, Y]_3$$

Then

$$\begin{aligned} [x * y - x *_{\infty} y] &= \left[ \frac{1}{2} [x_1, y_1]_3 \right] \lesssim (\|x_1\| \|y_1\|)^{1/3} \lesssim (\max(\|x_1\|, \|y_1\|))^{2/3} \\ &\leq [x]^{2/3} + [y]^{2/3}. \end{aligned}$$

This is Goodman's estimate with  $e = 2/3 = 1 - 1/s$ .  $\square$

## 5. SUBLINEAR BILIPSCHITZ EQUIVALENCE

**Definition 5.1.** Let  $X$  and  $Y$  be metric spaces; fix basepoints in  $X$  and  $Y$  and denote distances in  $X$  and  $Y$  by  $d$  and distances to basepoint by  $|\cdot|$ . Let  $\gamma \in [0, 1)$ . Let  $f: X \rightarrow Y$ . We say that  $f$  is a  $O(r^e)$ -sublinear bilipschitz equivalence if for some (any) left-invariant proper roughly geodesic distances on  $G$  and  $H$ ,

$$\frac{1}{\lambda} d(x, y) - O(|x|^e + |y|^e) \leq d(f(x), f(y)) \leq \lambda d(x, y) + O(|x|^e + |y|^e)$$

and  $d(y, f(X)) = O(|y|^e)$ .

*Remark 5.2.* For  $e = 1$ , this is exactly the definition of a quasiisometry.

**Exercise I.** Let  $X, Y, Z$  be pointed metric spaces. Let  $f: X \rightarrow Y$  be a  $O(r^e)$ -SBE and let  $g: Y \rightarrow Z$  be a  $O(r^{e'})$ -SBE. Show that  $g \circ f: X \rightarrow Z$  is a  $O(r^{\max(e, e')})$ -SBE.

**Theorem 5.3** (Cornulier). *Let  $G$  and  $H$  be simply connected nilpotent Lie groups. If  $G_{\infty} \simeq H_{\infty}$ , then there exists  $e \in [0, 1)$  (depending on the pair  $(G, H)$ ) such that  $G$  and  $H$  are  $O(r^e)$ -SBE*

*Proof.* Equip the vector space  $V$  underlying  $\mathfrak{g}_{\infty}$  with the group law  $*_{\infty}$ , and, via the data of supplementary subspaces to the lower central series, with the group law  $*$  of  $G$ . Let  $d_{\infty}$  and  $d$  denote proper geodesic distances on  $V$  which are respectively  $*_{\infty}$  and  $*$ -invariant. Using successively Guivarc'h's, Goodman's, and then again Guivarc'h's estimates we get

$$\begin{aligned} d(x, y) &= d(0, -x * y) && \text{since } d \text{ is } * \text{-invariant} \\ &= \lambda |(-x) * y| + c && \text{by Theorem 4.2} \\ &\leq \lambda |(-x) *_{\infty} y| + \lambda |(-x) * y - (-x) *_{\infty} y| + c && \text{by Theorem 4.6} \\ &\leq \lambda^2 d_{\infty}(0, (-x) *_{\infty} y) + \lambda(|x|^e + |y|^e) + \lambda c + c && \text{by Theorem 4.2} \\ &= \lambda^2 d_{\infty}(x, y) + O(|x|^e + |y|^e). && d_{\infty} \text{ is } *_{\infty} \text{-invariant} \end{aligned}$$

A similar argument shows that  $d_{\infty}(x, y) \leq \lambda^2 d(x, y) + O(|x|^e + |y|^e)$ . This proves that  $G$  and  $G_{\infty}$  are  $O(r^e)$ -sublinear bilipschitz equivalent. Similarly,  $H$  and  $H_{\infty}$  are  $O(r^{e'})$ -sublinear bilipschitz equivalent.  $\square$

Using Cornulier's theorem we can recover (a weak form of) the main result of Pansu's 1982's "first thesis" [Pan83]<sup>3</sup>.

<sup>3</sup>Pansu's first thesis is the "thèse de troisième cycle" which corresponds to today's doctorate. The second thesis (defended in 1987) that we evoke after is the "thèse d'état" which corresponds to today's habilitation.

**Definition 5.4.** Say that a sequence of compact metric spaces  $(X_n)$  **converges in Gromov-Hausdorff's sense** to a metric space  $X_\infty$  if

$$\inf_{\iota, \iota'} d_H(\iota(X_n), \iota'(X_\infty)) \rightarrow 0$$

where the infimum is taken over all isometric embeddings  $\iota, \iota'$  of  $X_n$  and  $X_\infty$  in compact metric spaces  $Z$ .

**Proposition and Definition 5.5** (Consequence of Gromov's compactness theorem). *Let  $G$  be a simply connected nilpotent Lie group, equipped with a left-invariant Riemannian metric  $d$ . There exists a proper metric space  $(X_\infty, \Delta)$  such that for all  $r > 0$ , the sequence of metric spaces*

$$X_n = \left( B_d(1_G, rn), \frac{d}{n} \right).$$

*has a subsequence that converges to  $B_\Delta(X_\infty, r)$  in Gromov-Hausdorff's sense. We call  $X$  a (pointed) **asymptotic cone** of  $X$ .*

By proper metric space we mean that the closed balls are compact. The proof (that we skip) is not very hard, but it is instructive; among the main ingredients of Gromov's proof, this is arguably the easiest to convey. See [Gro99]. Using a diagonal extraction we can assume that the same subsequence converges in Gromov-Hausdorff sense for all  $r$ .

*Remark 5.6 (\*)*. Those who know the subject may be aware of the following facts that are not present in our exposition.

- (1) There is a way to define asymptotic cones avoiding the extraction of subsequences, using a non-principal ultrafilter. This was realized by van den Dries and Wilkies [vdDW84] and then adopted by Gromov and geometric group theorists [Gro93], cf. Anthony's second talk on Tuesday.
- (2) It follows from the actual version of Pansu's theorem that asymptotic cones of nilpotent Lie groups are *unique* and that there is no need of passing to a subsequence. We don't want to use this because our current goal is to prove a weak version of Pansu's theorem. We will come back to this a bit later.

**Lemma 5.7.** *Let  $G$  and  $H$  be simply connected nilpotent groups. If there exists a  $O(r^e)$ -sublinear bilipschitz equivalence  $f: X \rightarrow Y$  for some  $e \in [0, 1)$ , then for any pointed asymptotic cone  $X_\infty$  of  $X$  there is a pointed asymptotic cone  $Y_\infty$  and a bilipschitz homeomorphism  $X_\infty \rightarrow Y_\infty$ .*

*Proof.* Consider

$$f_n: X_n \longrightarrow \left( H, \frac{d_H}{n} \right).$$

and let  $x_1, x_2 \in X_n$ . Then

$$\frac{d_H}{n}(f(x_1), f(x_2)) \leq \lambda \frac{d_G}{n}(x_1, x_2) + 2 \frac{(rn)^e}{n}.$$

Using the lower estimate on  $d(f(x_1), f(x_2))$  shows that  $X_n$  and  $f_n(X_n)$  are  $(\lambda, cn^{1-e})$ -quasiisometric. An Ascoli-Arzelà type argument finishes the proof.  $\square$

Note that the lemma holds with SBE replaced by quasiisometry. If  $f$  is a quasiisometry, it induces bilipschitz homeomorphisms between pairs of asymptotic cones

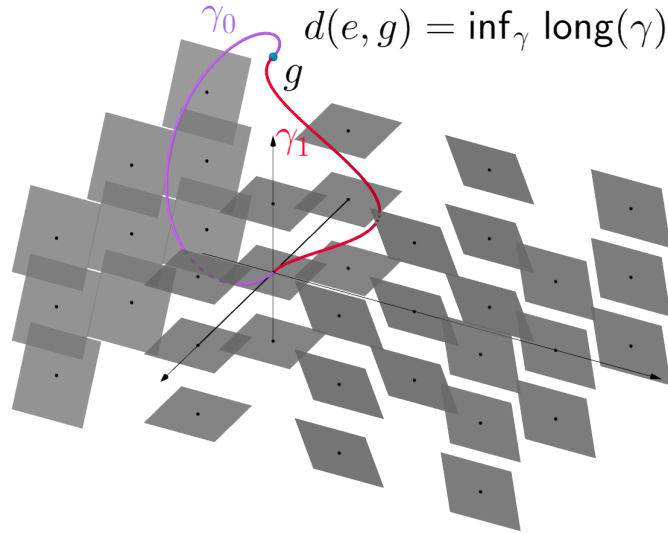


FIGURE 2. The Carnot-Carathéodory distance. The rectangles represent the horizontal distribution.

with *moving observation centers*. We first state the weak version of Pansu’s theorem and then explain the new words inside it.

**Theorem 5.8** (Weak version of “Pansu’s first thesis”). *Let  $G$  be a simply connected nilpotent Lie group. Any asymptotic cone of  $G$  is bilipschitz homeomorphic to  $G_\infty$  endowed with a Carnot-Carathéodory distance.*

We now explain what a Carnot-Carathéodory distance is. Let  $\mathfrak{g}$  be a nilpotent Lie algebra. Fix a norm  $\| - \|$  on  $\mathfrak{g}_1 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$  and equip the Carnot group  $G_\infty$  with the following distance (where CC stands for Carnot-Carathéodory): for every  $x, y \in \mathfrak{g}_\infty$ ,

$$(5.1) \quad d_{\text{CC}}(x, y) = \inf \int \|L_{\gamma(t)}^* \gamma'(t)\| dt,$$

the infimum being taken on piecewise  $C^1$ -paths<sup>4</sup> from  $x$  to  $y$  that such that  $\gamma'(t) \in (L_{\gamma(t)})_* \mathfrak{g}_1$  for all  $t$  (See figure 2). One can show that  $d_{\text{CC}}$  is a left-invariant, proper geodesic metric on  $G$  (see [LD25]). Thus  $d_{\text{CC}}$  can play the role of a metric  $d_\infty$  appearing in the proof of Theorem 5.3.

**Lemma 5.9.** *Let  $G$  be a Carnot group equipped with a Carnot-Carathéodory metric  $d_{\text{CC}}$ . For all  $t \in \mathbf{R}_{>0}$  and  $x, y \in G$ ,*

$$(5.2) \quad d_{\text{CC}}(t \bullet x, t \bullet y) = t d_{\text{CC}}(x, y)$$

*Proof.* The proof is not hard once we give a name to the relevant object. This object is the subbundle  $\mathcal{H}$  whose fiber above  $x$  is  $(L_{\gamma(t)})_* \mathfrak{g}_1$ . The dilation  $t \bullet -$  naturally extends to this bundle and sends  $\mathcal{H}_x$  into  $\mathcal{H}_{t \bullet x}$  multiplying norms by  $t$ . Especially,

<sup>4</sup>It is not very natural to take piecewise  $C^1$ -paths (and this certainly does not ease the proof of the claims we make here). However, it does not make a difference in the end (for non-trivial reasons, see [LD11, Theorem 1.2]) and it is simpler for the definition.

whenever  $\gamma$  is a curve between  $x$  and  $y$  that is tangent to  $\mathcal{H}$ ,  $t \bullet \gamma$  is a curve between  $t \bullet x$  and  $t \bullet y$  and it has length

$$\int \|L_{t \bullet \gamma(t)}^*(t \bullet \gamma)'(t)\| dt = t \int \|L_{\gamma(t)}^* \gamma'(t)\| dt.$$

This finishes the proof.  $\square$

**Lemma 5.10.** *Let  $G$  be a Carnot group. Any asymptotic cone of  $d_{CC}$  is isometric to  $d_{CC}$ .*

*Proof.* By the previous lemma,  $x \mapsto t \bullet x$  is a dilation of the Carnot-Carathéodory distance  $d_{CC}$ . It follows that all the spaces

$$X_n = \left( B_{d_{CC}}(1_G, rn), \frac{d_{CC}}{n} \right)$$

are all isometric, and at zero Gromov-Hausdorff distance. It follows that they are equal to any Gromov-Hausdorff limit of a subsequence.  $\square$

We can now assemble all the ingredients and prove the theorem.

*Proof of Theorem 5.8.* By Theorem 5.3,  $G$  and  $G_\infty$  are  $O(r^e)$ -sublinear bilipschitz equivalent. Let  $X_\infty$  be an asymptotic cone of  $G$ . By Lemma 5.7, there is some asymptotic cone of  $H$  to which  $X_\infty$  is bilipschitz homeomorphic.  $\square$

**Theorem 5.11** (One part of Pansu's second thesis from the SBE viewpoint). *If there exists  $e \in [0, 1)$  such that  $G$  and  $H$  are  $O(r^e)$ -SBE, then  $G_\infty \simeq H_\infty$ .*

*A few words on the proof.* (\*) First, note that we may assume without loss of generality that  $G$  and  $H$  are Carnot, replacing them with their associated Carnot groups if needed.

The heart of the argument is based on the observation that the dilation of a Carnot group is in many ways similar to the scalar dilation of a euclidean space. We have seen that this distance has the following fundamental property.

$$(5.3) \quad d_{CC}(t \bullet x, t \bullet y) = t d_{CC}(x, y)$$

for every  $t \in \mathbf{R}_{>0}$ . (In euclidean space, this corresponds to the somehow childish observation that scalar multiplication multiplies distance by a fixed amount.) Say that a map  $\phi: G \rightarrow H$  is P-differentiable at  $p \in G$  if

$$\forall y \in G, \lim_{t \rightarrow 0} D_P \phi_x(y) := \frac{1}{t} \bullet (\phi(x * t \bullet y) \phi(x)^{-1})$$

exists for all  $y \in G$  and that the convergence to the limit above is uniform on every compact set of  $G$ . If  $G$  and  $H$  are abelian and  $\phi$  is continuously differentiable, then  $D_P \phi_x$  is just the usual differential of  $\phi$  at  $x$ . Moreover, the differential  $D_P \phi_x$  is a Lie group homomorphism intertwining the Carnot dilations, very much like the usual differential of a differentiable map at a point is a linear map on the tangent space.

**Theorem 5.12** (Pansu's Rademacher theorem, [Pan89]). *Any lipschitz map between  $d_{CC}$  distances on Carnot groups is almost everywhere (with respect to a Haar measure) P-differentiable.*

The differential of  $\text{Cone}(f): G \rightarrow H$  is almost everywhere a (Carnot) Lie group isomorphism. It follows that  $G$  and  $H$  are isomorphic.  $\square$

*Remark 5.13* (\*). Pansu’s Rademacher theorem was later extended by Margulis-Mostow to cover more general subriemannian manifolds [Mar70]. Their proof is different (even just in the case of Carnot groups).

Let us finish our discussion of asymptotic cones of nilpotent Lie groups by coming back on the uniqueness issue. Here is a stronger form of Theorem 5.3.

**Theorem 5.14** (Pansu). *Let  $G$  be a simply connected nilpotent Lie group. The asymptotic cone of any proper left-invariant geodesic distance  $d$  on  $G$  is  $G_\infty$  endowed with a Carnot-Carathéodory distance.*

This theorem states that the asymptotic cone is unique. (The detailed version also says which norm one should put on  $\mathfrak{g}_1$ .) We want to point out that the uniqueness is an essential difference with Theorem 5.3. In fact this uniqueness, together with the two results below and Pansu’s differentiation theorem, are sufficient to deduce the slightly stronger statement.

**Fact 5.15** (Gromov<sup>5</sup> [Gro81]). *The asymptotic cone of a simply connected nilpotent Lie group is a locally compact group, equipped with a geodesic metric.*

**Fact 5.16** (Le Donne [LD15, Theorem 1.1]). *Carnot groups with  $d_{CC}$  metrics are the only metric spaces that are locally compact, geodesic, isometrically homogeneous, and admit a 1-parameter subgroup of metric dilations.*

In the statement of Fact 5.16, “isometrically homogeneous” means that there is a transitive subgroup of isometries. We warn the reader that although Le Donne’s proof of Fact 5.16 is quite short, it uses Montgomery-Zippin-Gleason’s solution to Hilbert’s fifth problem.

*Proof of Theorem 5.14 admitting the uniqueness of asymptotic cone of  $(G, d)$ .* We already know that some asymptotic cone of  $(G, d)$  is bilipschitz homeomorphic to  $(G_\infty, d_{CC})$ . Let  $X_\infty$  be that asymptotic cone. By uniqueness of  $X_\infty$ , it has a one-parameter of dilations. By Fact 5.15 and Le Donne’s characterisation of Carnot groups,  $X_\infty$  is some  $(H_\infty, d_{CC})$ . By Theorem 5.11  $G_\infty$  and  $H_\infty$  are isomorphic as Lie groups. Hence the asymptotic cone of  $(G, d)$  is  $(G_\infty, d_{CC})$  for some  $d_{CC}$ .  $\square$

## 6. A QUANTITATIVE PERSPECTIVE

The open problem from the beginning is not solved. But along the way, we have seen that some non-isomorphic nilpotent Lie groups are SBE. This leaves some room for the following.

**Imprecise question:** Given two simply connected, non-isomorphic nilpotent groups, to what extent are they not quasiisometric?

This is a softer question on which we can hope to make some gradual progress. Pansu and Cornulier’s theorems suggest to make the imprecise question precise in the following way.

<sup>5</sup>This is one step in the proof of the polynomial growth theorem and it is originally concerned with finitely generated groups, but there is no significative difference.

**Question 6.1.** Let  $N$  and  $N'$  be simply connected nilpotent groups with  $N_\infty \simeq N'_\infty$  as Lie groups. What is the range of  $e \in [0, 1)$  such that  $N$  and  $N'$  are  $O(r^e)$ -SBE?

**Definition 6.2.** Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be nilpotent Lie algebra, and let  $\phi : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{h})$  be a Lie group homomorphism. We denote  $\mathfrak{g} \times_\phi \mathfrak{h}$  and call central product of  $\mathfrak{g}$  and  $\mathfrak{h}$ , the Lie algebra

$$\mathfrak{g} \times_\phi \mathfrak{h} := (\mathfrak{g} \times \mathfrak{h})/\mathfrak{w},$$

where  $\mathfrak{w} = \{(X, \phi(X)) : X \in Z(\mathfrak{g})\}$ .

**Example 6.3.** Recall from Example 2.3 that

$$\mathfrak{g}_{4,3} = \text{span}(X_1, X_2, X_3, Y_1, Y_2, Z).$$

We have  $\mathfrak{g}_{4,3} = \mathfrak{l}_4 \times_\phi \mathfrak{l}_3$ , where  $\phi : Z(\mathfrak{l}_4) = \langle X_4 \rangle \rightarrow Z(\mathfrak{l}_3) = \langle X_3 \rangle$ ,  $\phi(X_4) = -X_3$ .

**Proposition and Definition 6.4.** Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be nilpotent Lie algebras with one-dimensional centers. Assume that  $\mathfrak{g}$  or  $\mathfrak{h}$  is Carnot. The isomorphism class of  $\mathfrak{g} \times_\phi \mathfrak{h}$  does not depend on the choice of a linear isomorphism  $\phi : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{h})$ . We denote it  $\mathfrak{g} \times_Z \mathfrak{h}$ .

*Proof.* Without loss of generality,  $\mathfrak{h}$  is Carnot; let  $s$  be its nilpotency class. Let  $\phi_1$  and  $\phi_2$  be isomorphisms from  $Z(\mathfrak{g})$  to  $Z(\mathfrak{h})$ . Because these are 1-dimensional, there is  $\alpha \in \mathbf{R} \setminus \{0\}$  such that  $\phi_2 = \alpha\phi_1$ . First assume that  $\alpha > 0$ , and let  $t = \alpha^{1/s}$ . Denote  $\mathfrak{w}_i = \{(X, \phi_i(X)) : X \in Z(\mathfrak{g})\} \subseteq \mathfrak{g} \times \mathfrak{h}$  for  $i = 1, 2$ . Define

$$\begin{aligned} \Psi : \mathfrak{g} \times_{\phi_1} \mathfrak{h} &\rightarrow \mathfrak{g} \times_{\phi_2} \mathfrak{h}, \\ (X, Y) \text{ mod. } \mathfrak{w}_1 &\mapsto (X, t \bullet Y) \text{ mod. } \mathfrak{w}_2. \end{aligned}$$

This is a Lie algebra isomorphism (technically, the map  $(X, Y) \mapsto (X, t \bullet Y)$ , which is clearly an isomorphism, descends to a well-defined morphism between the quotients; the reader can draw the commuting diagram to convince themselves). If  $\alpha < 0$  and  $s$  is odd, one can still define this isomorphism. If  $\alpha < 0$  and  $s$  is even: note that  $\mathfrak{h}$  admits a Lie algebra automorphism  $\varepsilon$  given by  $X \mapsto (-1)^i X$  for  $X \in \mathfrak{h}_i$ . One then gets an isomorphism after sending  $(X, Y) \text{ mod. } \mathfrak{w}_1$  to  $(X, \frac{1}{t} \bullet \varepsilon(Y)) \text{ mod. } \mathfrak{w}_2$ .  $\square$

Given a simply connected nilpotent Lie group  $G$  denote by  $\underline{e}(G)$  the infimal exponent  $e$  such that  $G$  and  $G_\infty$  are  $O(r^e)$ -SBE.

**Theorem 6.5** (Combination of [Cor19] and [GMLIP23]). *For every  $s \geq 3$*

$$\frac{1}{2s} \leq \underline{e}(L_s \times_Z L_3) \leq \frac{2}{s-1}.$$

The upper inequality is proved in [Cor19, Section 6]. The lower inequality involves the computation of Dehn function of the groups  $L_s \times_Z L_3$ . A large part of the methods of [GMLIP23] were developed in [LIPT23].

## APPENDIX A. LATTICES IN NILPOTENT LIE GROUPS

In the motivational §0 we have seen that one can associate a simply connected nilpotent Lie group (the connected nilshadow) to every compactly generated locally compact group of polynomial growth. This applies in particular to finitely generated groups of polynomial growth, which are virtually nilpotent by Gromov's theorem. In this case, the construction of the connected nilshadow proceeds from a construction of Mal'cev that we evoked in the end of §3 and it is much older than Losert's theorem. The aim of this appendix is to provide some details on this construction of Mal'cev.

We want to emphasize in particular that there are (many) simply connected nilpotent Lie groups without lattices, and that the QI classification problem is indeed more vast when posed for nilpotent Lie groups rather than when it is posed for finitely generated groups of polynomial growth (we will recall a proof of this based on a Baire category argument).

In this appendix, the Lie algebras can have a ground field different from  $\mathbf{R}$  (but always of characteristic 0). We assume more knowledge on Lie groups and Lie algebra than in the rest of the text.

**Zariski density of lattices.** We refer to Chapter II of Raghunathan's book [Rag72] for the proofs of the following results. In nilpotent Lie groups, all lattices are uniform<sup>6</sup>.

**Definition A.1.** Let  $G$  be a nilpotent Lie group and  $E \subset G$  a closed subset. We say that  $E$  is **Zariski-dense** in  $G$  if there exists a faithful unipotent representation  $\rho : G \rightarrow \mathrm{GL}(n, \mathbf{R})$  such that  $\rho(E)$  and  $\rho(G)$  have the same Zariski closure in  $\mathrm{GL}(n, \mathbf{C})$ .

A nilpotent Lie group  $G$  admits faithful unipotent representations as soon as it is simply connected. In this case, it is shown that it is equivalent, in the previous definition, to require that  $\rho(E)$  and  $\rho(G)$  have the same Zariski closure for any such representation  $\rho$ . Moreover,

**Theorem A.2** ([Rag72, th 2.1]). *For a closed subgroup  $\Gamma$  of a simply connected nilpotent Lie group  $G$ , the following are equivalent:*

- (i)  $\Gamma$  is Zariski-dense.
- (ii)  $G/\Gamma$  is compact.
- (iii)  $G/\Gamma$  admits a finite left-invariant measure.
- (iv)  $G$  is minimal among its closed connected subgroups containing  $\Gamma$ .

This theorem has several important corollaries. For example, [Rag72, cor 2.3], if  $\Gamma$  is a discrete subgroup of  $G$ , it is finitely generated, its rank is well defined, and the dimension of its Zariski closure  $\tilde{\Gamma}$  in  $G$  is equal to it [Rag72, th 2.10]. In particular, if  $\Gamma$  is a lattice, its rank is the dimension of the ambient group.

Finally, homomorphisms from the group are entirely determined by their restriction to a lattice, and there is no restriction on this:

**Proposition A.3** (strong rigidity). *Let  $G$  and  $G'$  be simply connected nilpotent Lie groups, and let  $\Gamma < G$  be a lattice. Every continuous homomorphism  $\Gamma \rightarrow G'$  extends to a unique global homomorphism. In particular, every automorphism of  $\Gamma$  extends to a unique automorphism of  $G$ .*

*Remark A.4* (\*). The uniqueness in the previous proposition is related to the Zariski-dense character of  $\Gamma$ . Let us detail an example to indicate how different the situation is in the semisimple case. Let the group  $G = \mathrm{SL}(2, \mathbf{R})$ . Let  $\Sigma$  be a hyperbolic surface of genus  $g \geq 2$ , and  $\Gamma = \pi_1 \Sigma$ . The space of faithful representations of  $\Gamma$  in  $G$  modulo conjugation is that of hyperbolic structures on  $\Sigma$ ; it is of dimension  $6g - 6$  according to Fenchel-Nielsen theory, hence much larger than  $\mathrm{Aut}(\mathrm{SL}(2, \mathbf{R}))$ . On the other hand, the group  $\mathrm{SL}(2, \mathbf{Z})$  is Zariski-dense in  $G$ , although it is not even cocompact, unlike  $\Gamma$ .

**Corollary A.5** (Translation in terms of commensurability). *Let  $\Gamma_1$  and  $\Gamma_2$  be two lattices in a simply connected nilpotent Lie group  $G$ . The following are equivalent:*

<sup>6</sup>This holds more generally in all locally compact groups that are amenable, [Gui73, lemma 1.10].

- (i)  $\Gamma_1$  and  $\Gamma_2$  admit finite index subgroups that are isomorphic:  $\Gamma_1$  and  $\Gamma_2$  are commensurable, [dlH00, IV.27].
- (ii) There exists  $g \in G$  such that  $g\Gamma_1g^{-1} \cap \Gamma_2$  is of finite index in both  $\Gamma_1$  and  $\Gamma_2$ :  $\Gamma_1$  and  $\Gamma_2$  are commensurable in  $G$ , [dlH00, II.28].

We follow the terminology of [dlH00], but the reader should beware that groups such that in (i) are also commonly called **abstractly commensurable**, while groups like in (ii) are commonly called just **commensurable**. In general it matters because these are different notions.

### Rational forms, lattices, and nilmanifolds.

**Theorem A.6** (Mal'cev 1949). *Let  $G$  be a real nilpotent, simply connected Lie group, with Lie algebra  $\mathfrak{g}$ . There are bijections between*

- (i)  $G$ -nilmanifolds, up to homeomorphism and finite covering of the same nilmanifold.
- (ii) Lattices of  $G$ , modulo the commensurability relation.
- (iii) Rational forms of  $\mathfrak{g}$ , i.e., isomorphism classes of  $\mathbf{Q}$ -Lie algebras  $\mathfrak{g}_{\mathbf{Q}}$  such that  $\mathfrak{g} \simeq \mathfrak{g}_{\mathbf{Q}} \otimes \mathbf{R}$ .

*In particular, for a simply connected Lie group  $G$  to have a lattice, it is necessary and sufficient that its Lie algebra  $\mathfrak{g}$  admits an  $\mathbf{R}$ -basis in which the structure constants are rational.*

Let us work out the case of the real Heisenberg group. The algebra  $\mathfrak{heis}$  is the unique non-abelian nilpotent Lie algebra of dimension 3, regardless of the field of scalars. The Heisenberg group thus admits a unique class of lattices in the sense of (ii). All Heisenberg nilmanifolds of dimension 3 cover

$$V_1 = \text{Heis}_{\mathbf{R}}(3)/\text{Heis}_{\mathbf{Z}}(3);$$

these are torus bundles over the circle  $S^1$ , with parabolic transition matrix  $\begin{pmatrix} 1 & \pm n \\ 0 & 1 \end{pmatrix}$  if  $n$  is the degree of the covering  $V_n \rightarrow V_1$ . (More generally, every nilmanifold is the total space of iterated torus bundles.)

*Remark A.7* (\*). Bridson, de la Harpe, and Kleptsyn show that, once equipped with the Chabauty topology, the space of lattices  $\mathcal{R}(\text{Heis}_{\mathbf{R}}(3))$  is a 6-dimensional manifold; it is itself a countable union (indexed by the integer  $n \geq 1$ ) of torus bundles over  $(S^3 - T) \times \mathbf{R}$ , where  $T$  is a trefoil knot in  $S^3$  [BdlHK09] (the factor  $S^3 - T$  in the base is  $\text{SL}_2(\mathbf{R})/\text{SL}_2(\mathbf{Z})$ , the space of lattices of covolume 1 in the plane  $\mathbf{R}^2 \simeq \text{Heis}_{\mathbf{R}}(3)^{\text{ab}}$ , see [BPS07, p.80].  $\mathcal{R}(\text{Heis}_{\mathbf{R}}(3))$  is a dense open set<sup>7</sup> in the Chabauty space of the real Heisenberg group.

**Example A.8** (Nilpotent Lie group without a lattice). There exist real nilpotent Lie algebras without a rational form. In fact, this is the case for most of them in high dimensions. Here is an argument for this [Rag72, 2.14] in degree 2, though it adapts to higher degrees, see [GMLIP23].

Let  $E$  (resp.  $V$ ) be a real vector space of dimension  $n$  (resp. 4). The nilpotent Lie algebra structures  $\mathfrak{g}$  of degree 2 on  $V \oplus E$  such that  $[\mathfrak{g}, \mathfrak{g}] = V$  are parameterized by

<sup>7</sup>A general fact in a nilpotent Lie group [BdlHK09, 3.4(iii)].

the set  $S$  of surjective<sup>8</sup>  $\alpha \in \Lambda^2 E \otimes V$ .  $S$  is an open subset of  $\Lambda^2 E \otimes V$ , which has dimension  $2n(n-1)$ . The group  $\mathrm{GL}(E) \times \mathrm{GL}(V)$  acts on  $S$  by

$$[(\varphi, \psi).s](u \wedge v) = \psi(s[\varphi^{-1}(u) \wedge \varphi^{-1}(v)]).$$

The orbits have dimension at most  $n^2 + 16$ ; however, once bases for  $V$  and  $E$  are fixed,  $\mathfrak{g}$  admits a rational structure if and only if its orbit contains an element with rational coordinates. By the Baire category theorem, as soon as  $\dim S > n^2 + 16$ , i.e., as soon as  $n \geq 6$ , these orbits cannot cover  $S$ .

*Remark A.9.* In dimension  $\leq 5$ , all real nilpotent Lie algebras admit at least one rational form [Cor16, p. 7].

**Pfaffian.** Theorem A.6 reduces, in principle, the classification of nilmanifolds covered by  $G$ , or of lattices in  $G$ , to algebra. However, this does not exhaust all the difficulty when it comes to obtaining an explicit classification. We will only treat a few examples in small dimensions. In this sense, it is useful to know how to associate invariants to isomorphism classes of rational Lie algebras, and then to determine which ones are realizable by the rational forms of a given real Lie algebra. The following construction comes from [Lau08]. Let  $\mathfrak{g}$  be a nilpotent Lie algebra of degree 2 over the field  $R$  (in the following,  $R = \mathbf{Q}$  or  $\mathbf{R}$ ). We denote by  $\mathrm{ad}^*$  its **coadjoint representation**, meaning

$$\langle \mathrm{ad}_X^* Y, Z \rangle = \langle Y, -\mathrm{ad}_X Z \rangle.$$

For every  $X \in \mathfrak{g}$ ,  $\mathrm{ad}_X^*$  is zero when restricted to  $[\mathfrak{g}, \mathfrak{g}]^\perp$ , so that  $\varphi \mapsto \{(X, Y) \mapsto \langle \mathrm{ad}_X^* \varphi, Y \rangle\}$  induces an endomorphism of  $\mathfrak{g}^*/[\mathfrak{g}, \mathfrak{g}]^\perp \simeq [\mathfrak{g}, \mathfrak{g}]^*$  to  $\mathfrak{g}^* \wedge \mathfrak{g}^* = \Lambda^2 \mathfrak{g}^*$ , which we denote by  $J$ . In fact,  $J\varphi$  vanishes on  $[\mathfrak{g}, \mathfrak{g}] \times [\mathfrak{g}, \mathfrak{g}]$  for every  $\varphi \in [\mathfrak{g}, \mathfrak{g}]^*$  since  $\mathfrak{g}$  is of degree 2, and thus  $J$  induces  $\bar{J} \in \mathrm{Hom}([\mathfrak{g}, \mathfrak{g}]^*, \Lambda^2 \mathfrak{g}_1)$ .

**Definition A.10** (Pfaffian). Assume  $\mathfrak{g}_1$  has even dimension  $2m$ , and denote by  $\mathcal{A}$  the  $K$ -algebra of polynomial functions of degree  $2m$  on  $[\mathfrak{g}, \mathfrak{g}]^*$ . Let  $\omega$  be a form of maximal degree on  $\mathfrak{g}_1$ . There exists a unique polynomial  $f$  on  $[\mathfrak{g}, \mathfrak{g}]^*$  such that

$$\forall \varphi \in [\mathfrak{g}, \mathfrak{g}]^*, \Lambda_k \bar{J}\varphi = f(\varphi)\omega.$$

Moreover, according to the Pfaffian theory, there exists  $\varepsilon \in \{\pm 1\}$  such that  $\varepsilon f$  is a square in  $\mathcal{A}$ . Let  $L$  be an extension of  $K$ . We denote by  $\mathrm{Pf}_L(\mathfrak{g})$  the equivalence class in  $\mathcal{A}$  of a square root of  $\varepsilon f$  under the action of  $K^* \times \mathrm{GL}_L([\mathfrak{g}, \mathfrak{g}]^*)$  given by

$$\forall P \in \mathcal{A}, (\lambda, g).P = \lambda P \circ g^{-1}.$$

We call it (or a representative) the **Pfaffian**.

**Example A.11.** Let  $\mathfrak{g} = \mathfrak{heis}_{2m+1}$  be the Heisenberg algebra of dimension  $2m+1$  over  $K$ , formed on  $(X_1, \dots, X_m, Y_1, \dots, Y_m, Z)$  with  $[X_i, Y_j] = \delta_{ij}Z$ . Then

$$\bar{J}[tZ^*](X_i, Y_j) = t\delta_{ij},$$

whence  $\mathrm{Pf}_K(\mathfrak{g}) = [t^m]$ .

The Pfaffian over the field of scalars is an invariant of the isomorphism class of  $\mathfrak{g}$ . Moreover, the Pfaffian behaves well under an extension of the ground field: if  $\mathfrak{g}_{\mathbf{Q}}$  is a rational form of  $\mathfrak{g}$ , then  $\mathrm{Pf}_{\mathbf{R}}(\mathfrak{g}) = \mathrm{Pf}_{\mathbf{R}}(\mathfrak{g}_{\mathbf{Q}})$ . Since the classification of forms (for example, quadratic forms) over  $\mathbf{Q}$  is considerably finer than that over  $\mathbf{R}$  (and a fortiori

<sup>8</sup> $\alpha$  defines a bracket by setting  $[u, v] = \alpha(\bar{u} \wedge \bar{v})$  and  $[u, [v, w]] = 0$  for all  $u, v, w \in V \oplus E$ , where  $\bar{x}$  denotes the projection of  $x$  onto  $E$  parallel to  $V$ .

over  $\mathbf{C}$ ), this does not a priori prevent the existence of a very large number of distinct rational forms for a given real or complex Lie algebra.

**An example: Lie algebras of type (4, 2).** Still following [Lau08], say that a Lie algebra  $\mathfrak{g}$  is of type (4, 2) if it is nilpotent of degree 2, and such that  $\dim \mathfrak{g}_1 = 4$  and  $\dim[\mathfrak{g}, \mathfrak{g}] = 2$ . In this case, the Pfaffian is a class of binary quadratic forms, and it is a complete invariant, by the following theorem.

**Proposition A.12** (Grunewald-Segal-Sterling, 1982). *Let  $\mathfrak{g}, \mathfrak{g}'$  be two Lie algebras of type (4, 2) over a field of characteristic zero, with the same Pfaffian. Then  $\mathfrak{g}$  and  $\mathfrak{g}'$  are isomorphic.*

Since binary quadratic forms are themselves classified by their discriminant, the previous proposition gives the classification of Lie algebras of type (4, 2)

(a) **over the rationals:** The rational Lie algebras of type (4, 2) are indexed by  $\mathbf{Q}/\mathbf{Q}^{\times 2}$  and a complete system of representatives is given by

$$\mathfrak{n}_k := \langle X_1, \dots, X_4, Z_1, Z_2 \rangle$$

with the Lie brackets

$$\begin{aligned} [X_1, X_3] &= [X_2, X_4] = Z_1, [X_1, X_4] = Z_2, \\ [X_1, X_2] &= [X_3, X_4] = 0, [X_2, X_3] = kZ_2. \end{aligned}$$

for every integer  $k$ , zero or square-free. We verify using the previous basis that

$$\frac{\Lambda_2 J(xZ_1^* + yZ_2^*)}{X_1^{\text{ab}} \wedge \dots \wedge X_4^{\text{ab}}} = \det \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & ky & x \\ -x & -ky & 0 & 0 \\ -y & -x & 0 & 0 \end{pmatrix} = (x^2 - ky^2)^2.$$

Hence  $\text{Pf}_{\mathbf{Q}}(\mathfrak{n}_k) \equiv x^2 - ky^2$ , with discriminant  $k$ .

(b) **over the reals:** The discriminant retains only the sign, and there are 3 isomorphism classes of Lie algebras of type (4, 2): for every  $k > 0$

$$\begin{aligned} \mathfrak{n}_- &\simeq \mathfrak{n}_k \otimes \mathbf{R} & k < 0 \\ \mathfrak{n}_0 &\simeq \mathfrak{n}_0 \otimes \mathbf{R} \\ \mathfrak{n}_+ &\simeq \mathfrak{n}_k \otimes \mathbf{R} & k > 0 \end{aligned}$$

We verify that  $\mathfrak{n}_1 \otimes \mathbf{R} = \mathfrak{heis} \oplus \mathfrak{heis}$  by setting  $Y_1^\pm = (X_1 \pm X_2)/2$ ,  $Y_2^\pm = (X_3 \pm X_4)/2$  and  $Z^\pm = (Z_1 \pm Z_2)/2$ , and by calculating  $[X^\pm, Y^\pm] = Z^\pm$  and  $[X^\pm, Y^\mp] = 0$ . Similarly,  $\mathfrak{n}_{-1} \otimes \mathbf{R}$  is the Lie algebra underlying  $\mathfrak{heis}(\mathbf{C})$ .

*Remark A.13.* We have given examples with zero, one, or infinitely many rational forms. According to [dlH00, p. 304], the question of whether there exists a real nilpotent Lie algebra with a finite number  $k \neq 1$  of rational forms is open.

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