Filling in nilpotent groups: old and new

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This is an AI-less talk.

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... but actually not every fox does that.

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From the fox I take the wit. I won't discuss ongoing work, but I will show some of my trails : **what I am thinking about**, not necessarily with a new result in mind.

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This has many variants and generalizations.

- What is a boundary ?

Lipschitz cycle $c = \sum \lambda_i c_i$ $c_i : \Delta^k \to \mathbb{R}^n$ $\partial c = 0$

- What is a (minimal) surface ? Do we care ? → Not in this talk

Surface : lipschitz chain C such that $\partial C = c$

- Does it make sense in metric spaces ? Yes

For us. Contractible Riemannian manifold

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The filling functions measure the least area needed to fill all boundaries of every given size.

(Homological) filling functions, formally

Let X be a contractible smooth n-dimensional Riemannian manifold of bounded geometry.
Let $k \in \{2, \dots, n\}$. Define

filling volume

$$\mathrm{FV}^k_X(v) = \sup_{c \in \mathrm{C}^{\mathrm{lip}}_{k-1}(X,\mathbf{Z}):\partial c = 0, |c| \leqslant v} \inf\{|r| : r \in \mathrm{C}^{\mathrm{lip}}_k(X,\mathbf{Z}) : \partial r = c\}$$

filling volume of c

$c = \sum \lambda_i c_i$

$|c| = \sum |\lambda_i| \mathrm{mass}(c_i)$

where $|\cdot|$ means "mass".

$\mathrm{mass}(c_i) = \int_{\Delta^k} |Dc_i|$

Filling functions and groups

Let G be a contractible Lie group of dimension n. Let $k \in \{2, \dots, n\}$. Define

$X = (G, d)$

$$\mathrm{FV}^k_G(v) := \mathrm{FV}^k_{(G,d)}(v),$$

where d is an invariant Riemannian metric on G.
This doesn't really depend on d.

d_1, d_2 two such metrics

d_1 and d_2 are bilipschitz: $\exists L \geq 1, \quad \frac{1}{L} d_1 \leq d_2 \leq L d_1$

$\mathrm{FV}^k_{(G,d_1)}(v) \approx \mathrm{FV}^k_{(G,d_2)}(v)$ where $f(v) \approx g(v)$ means $f(v) \leq C g(Cv)$, $g(v) \leq C f(Cv)$

(Growth types of) filling functions are well-defined for groups (with suitable finiteness props), and quasi-isometry invariants.

Homological and homotopical filling

X as before, $k \in \{2, \ldots, n = \dim X\}$.
Define $\delta_X^{k-1}(v)$ similarly as $\mathrm{FV}_X^k(v)$, replacing cycles with spheres and chains with balls. For groups $\delta^1 = \delta$ a.k.a. Dehn function.

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(k-1)

Proposition (Abrams-Brady-Dani-Young + earlier work)

For $k \geqslant 4$, δ_X^k and FV_X^k have same growth type.

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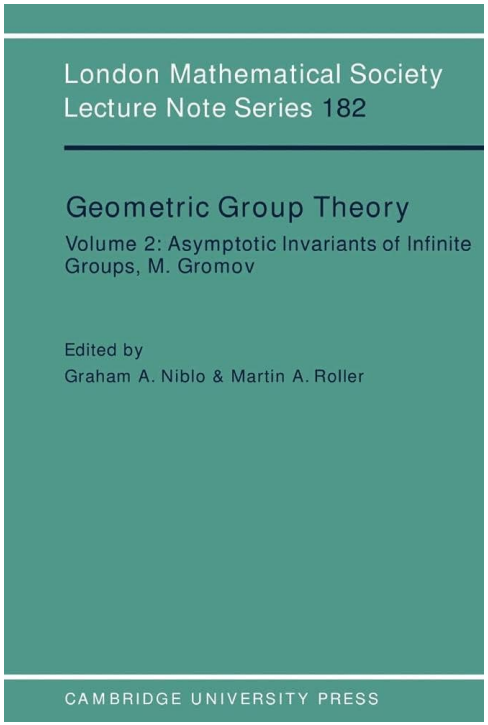
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Theorem (Abrams-Brady-Dani-Young 2013)

$\exists$ group H such that $\mathrm{FV}_H^2 \neq \delta$

An exercise from Gromov's UMO[1] *Asymptotic Invariants of Infinite Groups*



193 pages of exercises ranging from "standard" to "unsolved" as of today.

Theorem (Gromov 93)

Simply connected nilpotent Lie groups have polynomially bounded filling invariants.

(Proof given = exercise, a few hints.)

Can embed it as a closed subgroup of strictly upper-triangular nxn matrices

→ contractible

1. Unidentified Mathematical Object

Almost-solution : asymptotic cones

2 ingredients $\pi_{k-1}(\mathrm{Cone}_\omega(G)) = \{1\} \Rightarrow \delta_G^{k-1}$ is polynomially bounded

A. Theorem (Riley's thesis 2003, $k = 2$ Gromov, Drutu 90s)

If G has $(k-1)$-connected asymptotic cones, then δ_G^{k-1} is polynomially bounded. « picture of the group as seen from infinity »

B. Theorem (csq of Pansu's thesis 1983)

Nilpotent Lie groups have (subRiemannian) Carnot groups as asymptotic cones. Carnot groups are contractible.

for $k \geq 4$, $FV_G^k(v)$ is polynomially bounded

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$k = 2$ $FA_G(v) \preceq \widehat{\delta}$ ← polynomially bounded

$k = 3$ is missing

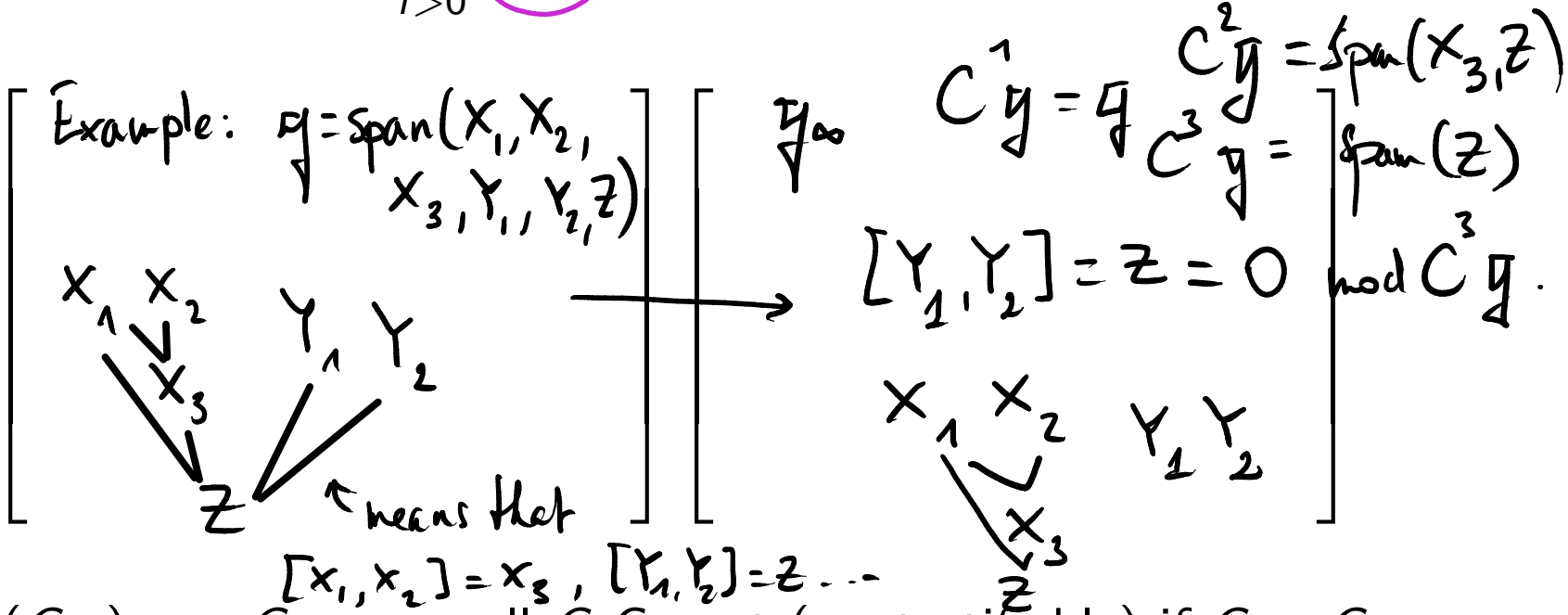
What is a Carnot group?

Descending central series: $C^1\mathfrak{g} = \mathfrak{g}$, $C^i\mathfrak{g} = [C^{i-1}\mathfrak{g}, \mathfrak{g}]$

Let $\mathfrak{g}$ be the nilpotent Lie algebra of G. Define its Carnotification

$$\mathfrak{g}_\infty = \bigoplus_{i>0} C^i\mathfrak{g}/C^{i+1}\mathfrak{g};\ G_\infty = \exp \mathfrak{g}_\infty.$$

Carnotification of G



1. $(G_\infty)_\infty = G_\infty$; we call G Carnot (or stratifiable) if $G \simeq G_\infty$.
2. subRiemannian G_∞ is the asymptotic cone of G (Pansu).
3. Not all nilpotent groups are Carnot!

Wenger's upper bound in Carnot groups

Theorem (Csq of Wenger 2011)

Let G be a Carnot group of nilpotency class c. Then for every k there is an (explicit) $e_k = e_k(c)$ such that

$$\mathrm{FV}^k_G(v) \preccurlyeq v^{e_k}.$$

Moreover,

- *for $k = 2$, $e_2(c) = c + 1$*
- *for $c = 2$, $e_k(2) \leq \frac{2^k - 1}{2^{k-1} - 1}$.*

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Sharp in general

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- *for $c = 2$, $e_k(2) \leq \frac{2^k - 1}{2^{k-1} - 1}$.* $\xrightarrow[k \to \infty]{}$ 2

Wenger's bound is a bit better in general.
$k = 2$ is sharp, also follows from Gersten-Holt-Riley 03 who proved $\delta_G(v) \preccurlyeq v^{c+1}$ when $\mathfrak{g}$ has structure constants in $\mathbf{Q}$ and not assumed Carnot (gen. case Llosa Isenrich-P-Garcia Mejia 23).

Heisenberg groups and their filling invariants

$\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or $\mathbf{O}$. Define $\mathfrak{heis}^n_{\mathbf{K}} = \mathbf{K}^{n-1} \oplus \mathfrak{Im}(\mathbf{K})$ ($n \leq 2$ if $\mathbf{K} = \mathbf{O}$),

$$[(x_1, \ldots, x_{n-1}, z), (y_1, \ldots, y_{n-1}, z')] = \frac{1}{2} \sum \mathfrak{Im}(x_i \overline{y_i})$$

and $\mathrm{Heis}^n_{\mathbf{K}} = \exp \mathfrak{heis}^n_{\mathbf{K}}$.

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For $\mathbf{K} = \mathbf{C}$ this is $2n - 1$-dim Heisenberg.

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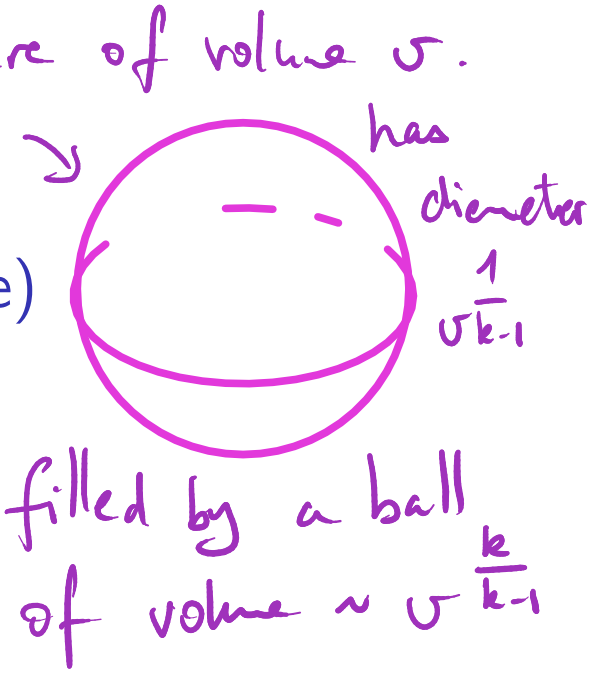
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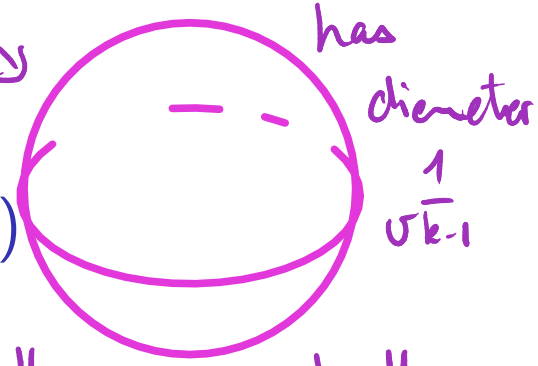
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a $(k-1)$-sphere of volume v.



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filled by a ball of volume $\sim v^{\frac{k}{k-1}}$

Theorem (Young 06+10 (conj. Gromov))

$$\mathrm{FV}^k_{\mathrm{Heis}^n_{\mathbf{C}}}(v) = \begin{cases} v^{\frac{k}{k-1}} & 2 \leq k \leq n-1 \\ v^{\frac{k+1}{k-1}} = v^{\frac{n+1}{n-1}} & k = n \\ v^{\frac{k+1}{k}} & n < k \leq 2n-1 \end{cases}$$

← Small degrees: Euclidean filling

← Super-Euclidean

Heisenberg groups and their filling invariants, ctd

Theorem (Gruber's thesis 2015, conjex. Gromov)

Euclidean in small degree

$$\mathrm{FV}^k_{\mathrm{Heis}^n_{\mathbf{H}}}(v) = v^{\frac{k}{k-1}}, 2 \leq k \leq n-1$$

For $\mathbf{K} = \mathbf{O}$ Wenger's and Gersten-Holt-Riley's upper bounds give :

- $\mathrm{FV}^2_{\mathbf{Heis}^2_{\mathbf{O}}}(v) \preccurlyeq v^3$ and
- $\mathrm{FV}^3_{\mathbf{Heis}^2_{\mathbf{O}}}(v) \preccurlyeq v^{7/3}$.

$\frac{2^3-1}{2^2-1}$

These estimates have applications to filling in lattices (discrete subgroups of finite covolume) in simple Lie groups of $\mathbf{R}$-rank one.

Application : filling in lattices in simple Lie grps of rank one

Let G be a simple Lie group of $\mathbf{R}$-rank one. Concretely
$G = \mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $F_4^{(-20)}$.
$N =$ max unipotent of a minimal parabolic in G. These are always Heisenberg groups.

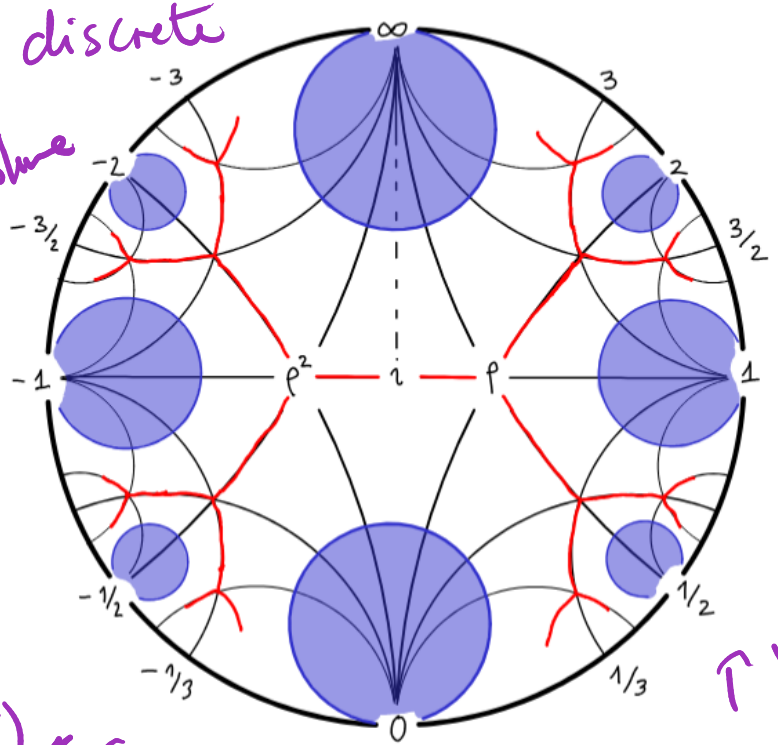
Application : filling in lattices in simple Lie grps of rank one

Let G be a simple Lie group of **R**-rank one. Concretely $G = \mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$ or $F_4^{(-20)}$.

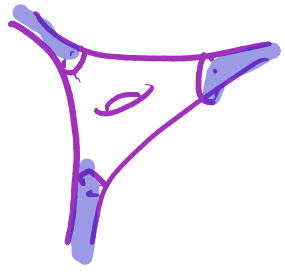
N = max unipotent of a minimal parabolic in G. These are always Heisenberg groups.

Γ = lattice in G. "Interesting" case is when G/Γ is noncompact.

$\Gamma < G$ discrete
$\Gamma \backslash G$ finite volume



A non-uniform lattice in G is quasi-isometric to its compact core in the symmetric space

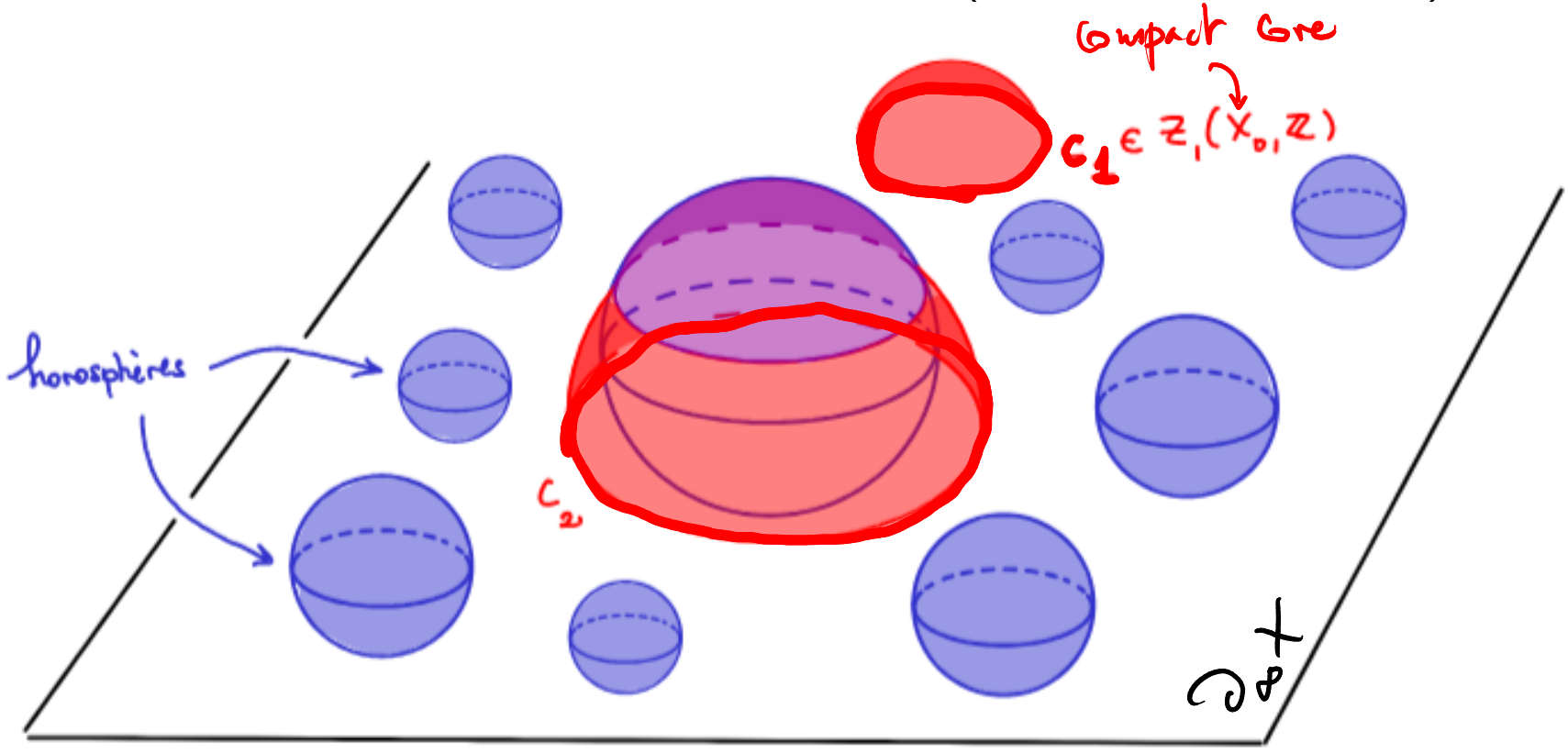


$\Gamma \backslash G$

$SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$

Filling in lattices in **R**-rank one, ctd.

Filling in the compact core of Γ is exactly as hard as filling in the cusp subgroups of Γ which are lattices in N (a Heisenberg group).



Top-dimensional filling

If G is simply connected nilpotent, the Bass-Guivarc'h formula (1973) implies that it has polynomial growth rate of order

$$N(G) = \sum_i \dim C^i G = \sum_i i \dim C^i G / C^{i+1} G.$$

This is also the Hausdorff dimension of the asymptotic cone.

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Theorem (Varopoulos + students Saloff-Coste and Coulhon)

Let G be n-dim 1-connected nilpotent group. Then

$$\mathrm{FV}_G^n(v) = v^{\frac{N(G)+1}{N(G)}}$$

mass transportation argument

(lower bound given by balls).

Sketch of a solution to Gromov's exercise

Seems not written before 2026 by non-Gromov people for $k = 3$ and G non-Carnot. This is medium hard.

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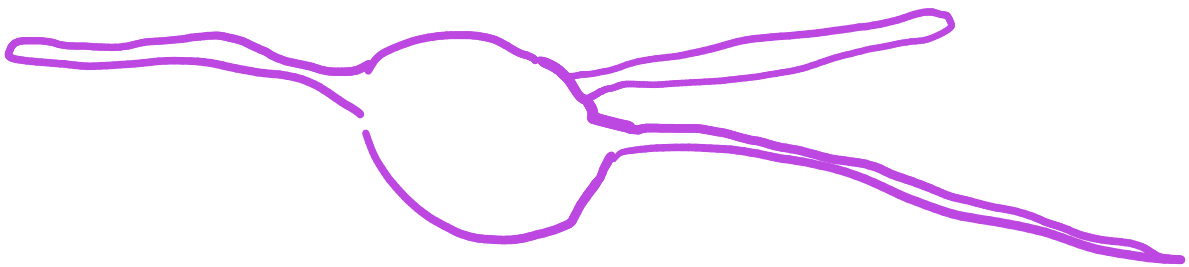
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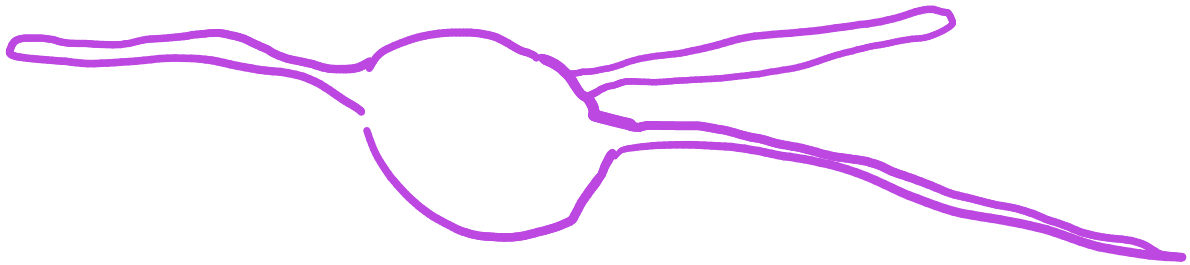
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4. Solution : use the Federer-Fleming deformation theorem to cut the (hypothetical) hair. Needs a uniform triangulation.

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Definition (Bader-Sauer 2023)

A locally compact second countable group G has property (T_n) if its continuous cohomology $H^k_c(G, \pi)$ vanishes for every unitary G-rep π without invariant vectors and every $k \in \{1, \dots, n\}$.

For $n = 1$ this is eq. to Kazhdan's property (Guichardet 70s).

Bader-Sauer prove that $\mathrm{SL}(n, \mathbf{Z})$ has (T_{n-2}).

Filling invariants and polynomial cohomology

Let $\Gamma = \mathrm{SL}(n, \mathbf{Z})$ and $G = \mathrm{SL}(n, \mathbf{R})$.

Theorem (Leuzinger-Young 2017)

Γ has Euclidean filling in all degrees $k < n - 1$.

Bader-Sauer deduce property (T_{n-2}) for Γ from G using polynomial cohomology and this theorem.

$$\begin{array}{ccc} H^k(\Gamma, \pi) & \xrightarrow{\simeq} & H_c^k(G, \mathrm{Ind}_\Gamma^G \pi) \\ \uparrow & & \uparrow \\ H^k_{\mathrm{pol}}(\Gamma, \pi) & \xrightarrow{\simeq} & H^k_{\mathrm{pol}}(G, \mathrm{Ind}_\pi^G) \end{array}$$

for $k < n-1$ this is an isomorphism

12 an isomorphism

Actually, Bader-Sauer only use that Γ has polynomial filling in degrees $k < n - 1$.

Lopez-Neumann–Paucar's polynomial cohomology

Lopez-Neumann and Paucar introduce a refinement of polynomial cohomology, taking care of the actual degree of growth of cochains. Under the assumption of polynomiality of filling invariants of a group Γ, they obtain isomorphisms of comparison maps

$$H^k_{\text{poly+params.}}(\Gamma, \pi) \to H^k(\Gamma, \pi)$$

for isometric reps. π of Γ on Banach spaces. The parameters for which there is an isomorphism depend crucially on the *product* of polynomial degree of filling functions up to order k,

$$N_k = \prod_{j=2}^{k} \deg(\mathrm{FV}^j_\Gamma)$$

Lattices in $F_4^{(-20)}$ have higher property T

In another life, $F_4^{(-20)}$ has higher rank. But how high ?
Bader-Sauer prove that it has[2] (T_3).

2. In fact it happens for other groups has well, for instance G_2^2 has real rank 2, yet it has (T_2) (but not (T_3)).

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Using this, together with the fact that Wenger's bound on the filling in the octonionic Heisenberg group ensures to them that the comparison map is an isomorphism, Lopez-Neumann and Paucar are able to deduce :

Theorem (Lopez-Neumann – Paucar, january 2026)

(Non-uniform) lattices Γ in the simple Lie group F_4^{-20} have property (T_3), and even slightly better, namely every unitary Γ-rep. has vanishing H^1, H^2 and H^3.

There are other applications, e.g. to vanishing of ℓ^p-cohomology for lattices in **R**-rank one groups.

---

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Another application, to L^p measure equivalence

Theorem (Lopez-Neumann – Paucar, december 2025)

Let Γ and Λ be two finitely generated nilpotent groups. There exists $p \in [1, +\infty)$ such that if Γ and Λ are L^p-measure equivalent, then $b_k(\Gamma) = b_k(\Lambda)$ for all k.

The best p in the theorem depends explicitely on the filling functions of Γ and Λ (very much unknown in general).

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For nilpotent groups, L^p-measure equivalence $\Leftarrow$ L^∞ measure equivalence $\iff$ quasiisometry. This theorem is a strengthening of Shalom's 2004 theorem that Betti numbers of nilpotent groups are quasiisometric invariants.

Thank you for your attention ! This was an AI-less talk.

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Soap bubbles in Prypiat