Regular polytopes with H_d symmetry

Johannes Kepler's mathematical stars

Gabriel Pallier

Outline

Background on convex and star polyhedra

Star polytopes and platonic Riemann surfaces

Schläfli-Hess polytopes

Convex regular polygons

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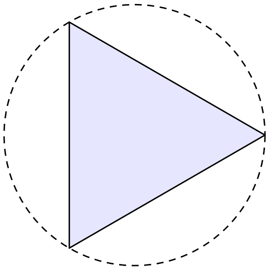

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Figure: The convex regular 3-gon.

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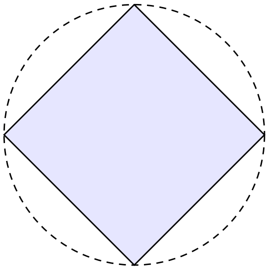

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Figure: The convex regular 4-gon.

Denote by $\{p\}$ the convex regular p-gon. The integer p is the number of 0-cells (vertices) and of 1-cells (edges).

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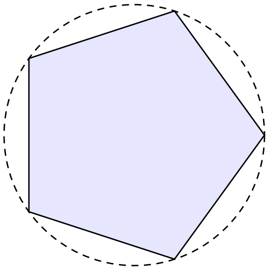

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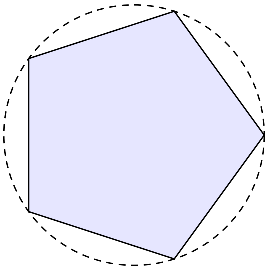

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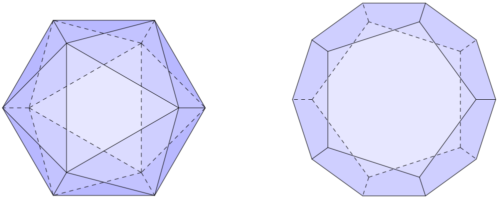

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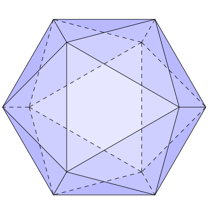

Figure: The regular icosahedron $\{3, 5\}$ and dodecahedron $\{5, 3\}$.

There exists finitely many such polyhedra: the five Plato solids.

Why finitely many ?

Equivalently, real convex polyhedra are regular tilings of the geometric sphere $\mathbb{S}^2$.

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$$\mathrm{Area}(\triangle) = \frac{\pi}{2} + \frac{\pi}{p} + \frac{\pi}{q} - \pi.$$

As $\mathrm{Area}(\triangle)$ must be nonnegative, a necessary (and in fact sufficient) condition for $\{p,q\}$ to define a convex polyhedron is

$$\frac{1}{p} + \frac{1}{q} > \frac{1}{2} \qquad \text{(spherical group eq.)}$$

Cell numbers

Using the expression for $\mathrm{Area}(\triangle)$ and Euler's formula, one can recover the full combinatorial data of $\{p, q\}$: denoting by $\{p, q\}^{(k)}$ the set of k-cells

$$\left|\{p,q\}^{(2)}\right| = \frac{4\pi}{2p \cdot \mathrm{Area}(\triangle)} = \frac{4q}{2p+2q-pq}, \qquad \text{(Conv 2)}$$

$$\left|\{p,q\}^{(0)}\right| = \left|\{q,p\}^{(2)}\right| = \frac{4p}{2p+2q-pq}, \qquad \text{(Conv 0)}$$

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where the last equality accounts for $\chi(S^2)=2$, after possibly triangulating the 2-cells.

Coxeter's kaleidoscope

A presentation for $\mathrm{W}_{p,q}$ is

$$\mathrm{W}_{p,q} = \left\langle r, s, t \mid r^2, s^2, t^2, (rs)^p, (st)^q \right\rangle . \qquad \text{(Cox)}$$

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Remark (1)

By Poincaré's observation on fundamental polygons, any discrete group generated by reflections on one of the model spaces $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ has presentation (Cox).

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Remark (2)

An important consequence of the representation in $\mathrm{O}(V)$ is that groups with (Cox) presentation are **virtually torsion-free** (i.e. contain finite index torsion-free subgroups). In general this is a consequence of Selberg's lemma.

Trichotomy of Coxeter complexes

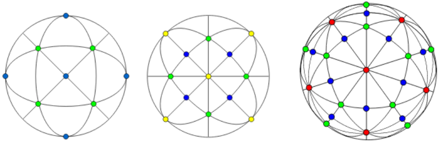

Figure: Spherical Coxeter complexes.

To produce a regular tesselation/polyhedron amounts to choosing a non right-angled vertex point in a Coxeter chamber.

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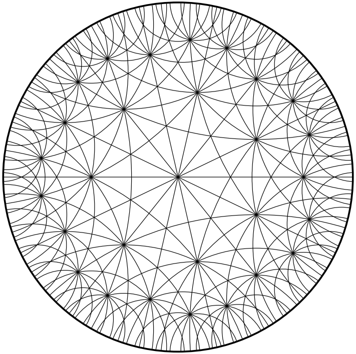

Figure: Part of Coxeter complex for the group $\mathrm{W}_{3,7}$

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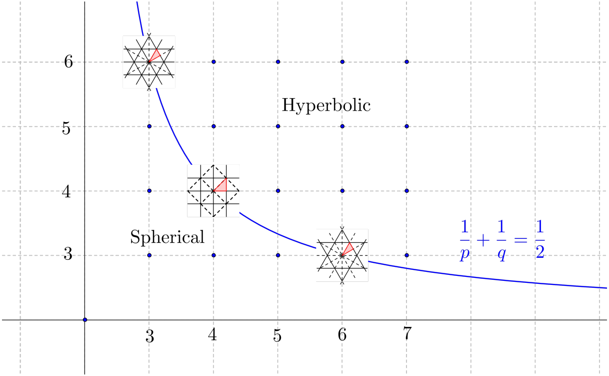

Figure: Affine Euclidean tesselation, with Coxeter chamber in red.

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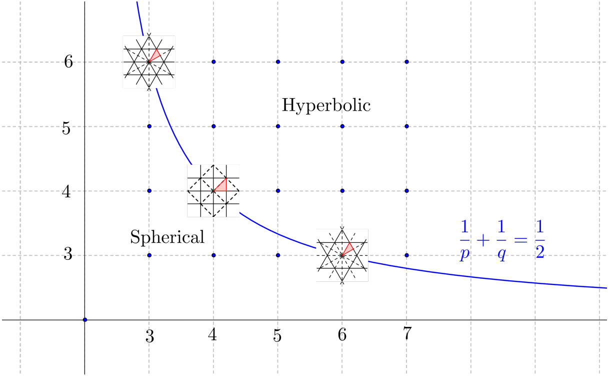

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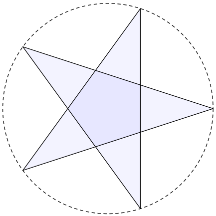

$$\left\{ \frac{5}{2} \right\}$$

Figure: Nonconvex regular polygons

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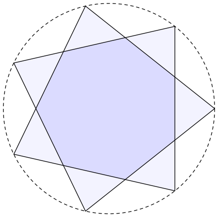

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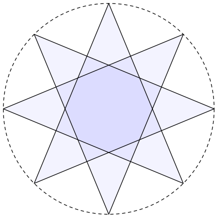

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Regular polygons and coverings of S^1

To each $\{p/d\}$, one can associate a topological covering of S^1, e.g. by projecting radially $\partial\{p/d\}$ onto $\partial\{p\} \simeq S^1$.

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Except for degree 1, the covering does not extend to a topological covering of the 2-cell $D \simeq B^2$ of $\{p\}$. Nevertheless, if D is promoted to the unit complex disk $\mathbb{D}$, then as $z \mapsto z^d$, it extends to a branched covering.

Finite regular polyhedra

Definition

An immersion ("geometric realization") of a finite abstract polyhedron is regular if

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2. *All vertex figures are realized as the same regular polygon $\{p_2/d_2\}$.*

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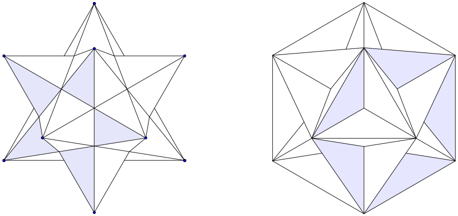

Figure: Edge-first views of the small stellated dodecahedron $\{5/2, 5\}$ and great dodecahedron $\{5, 5/2\}$, the first two stellations of a regular dodecahedron.

Star polyhedra

Apart from the 5 convex ones, there exists 4 nonconvex regular polyhedra. Kepler recognized two of them, plus the nonconnected stella octangula, in *Harmonices Mundi* (1619). At least $\{5/2, 5\}$ was actually known before Kepler. Poinsot isolated the four and called them regular ; the list was proved complete (in a sense) by Poinsot and Cauchy.

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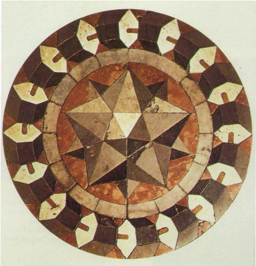

Figure: Marble floor of Basilica St Mark, Venice, circa 1430.

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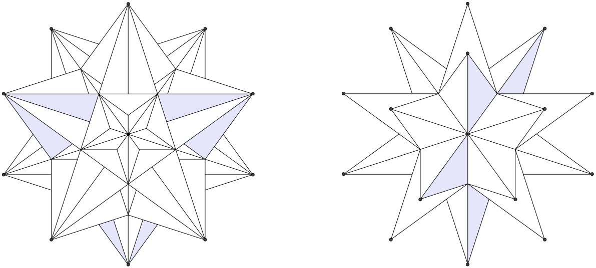

Figure: Vertex-first view of a great icosahedron $\{3, 5/2\}$ (on the left) and great stellated dodecahedron $\{5/2, 3\}$ (on the right).

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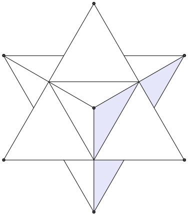


Figure: Vertex-first (and face-first) view of Kepler's *stella octangula*, stellation of the octahedron: a compound (i.e. disconnected union) of two tetrahedra.

Degree

As in the convex case, one can associate to any star polyhedron P a tiling of the sphere by right-angled triangles, this time with overlap. The number of overlapping triangles at a non-vertex point is called the degree of P. For instance, the small stellated dodecahedron $\{5/2, 5\}$ has degree 3.

Combinatorics of star polyhedra

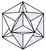
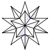
P	symbol	vertices	edges	faces	$\chi(P)$	$\deg(P)$
	$\{5/2,5\}$	12	30	12	-6	3
	$\{5,5/2\}$	12	30	12	-6	3
	$\{5/2,3\}$	20	30	12	2	7
	$\{3,5/2\}$	12	30	30	2	7
		8	12	8	4	2

Table: Combinatorics and degrees of star polyhedra (and a nonconnected intruder). Observe that for the stella octangula, $\chi(P) = \deg(P)\chi(S^2) = 4$. We shall give a subtler relation between χ and $\deg$ in the connected case.

Overview: Conway's hexagon

Several poyhedra in the preceding table have identical combinatorics. In fact one can check that pairs of opposite polyhedra in the following hexagon are isomorphic **as abstract polyhedra**.

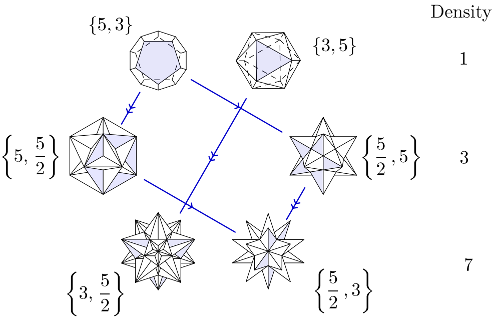


Figure: Hexagonal arrangement of the polyhedra with group H_3. Single arrows depicts *stellation* (extend 1-cells and fill), double arrow *greatening* (extend 2-cells).

Outline

Background on convex and star polyhedra

Star polytopes and platonic Riemann surfaces

Schläfli-Hess polytopes

Hyperbolic Riemann surface

Implicitly, all the surfaces are assumed connected.

Theorem and Definition

A hyperbolic compact Riemann surface is, equivalently:

1. *A 1-dimensional complex manifold, compact, such that the underlying topological surface has genus $\geqslant 2$.*
2. *A conformal class of Riemannian metrics on a compact smooth surface of genus $\geqslant 2$. This class has a preferred metric of Gauss curvature -1.*
3. *A compact quotient by a discrete, freely operating group Γ of automorphisms of $\mathbb{D}$ / isometries of $\mathbb{H}^2$, up to conjugacy.*

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Equivalence between the two formulations of item 3 can be seen as a consequence of the Schwarz-Pick lemma, while existence in 2 requires uniformization theorem.

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Any compact Riemann surface is either hyperbolic, or

- The Riemann sphere $\mathbb{C}\mathrm{P}^1$.
- $\mathbb{C}/\Lambda$ for $\Lambda < \mathbb{C}$ a lattice (equivalently, a flat metric on T^2).

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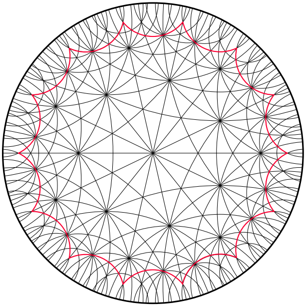

Figure: Fundamental domain for Γ' in $\mathbb{H}^2$.

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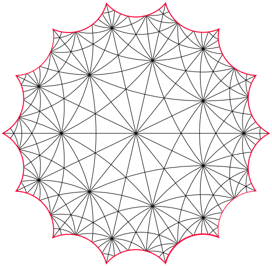

Figure: Fundamental domain for Γ', only.

The Riemann surface $X = \mathbb{H}^2/\Gamma'$ is the Klein quartic. It can be obtained by guing sides $2n$ and $2n+5$ of the 14-gon depicted in appropriate directions (that can be found by drawing the heptagons).

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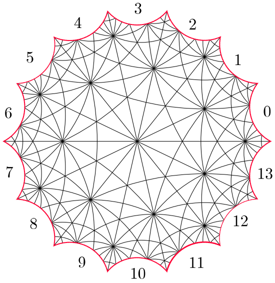

Figure: Label the edges of the 14-gon

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The Klein quartic is a platonic Riemann surface : it has a tesselation by polygons (heptagons or triangles, according to the vertex chosen in the Coxeter chambers) with automorphism group acting vertex- and face-transitively.

Holomorphic maps

Thanks to analyticity properties of holomorphic maps, morphisms in the category of compact Riemann surfaces behave nicely:

Proposition

Let $f : X \to Y$ be a holomorphic map between compact connected Riemann surfaces. Then there exists finite sets $R \subset X$, $f(R) = S \subset Y$, such that $f : X \setminus R \to Y \setminus R$ is a n-sheeted topological covering for some n, and for all $Q \in Y$,

$$\sum [f(P) = Q] e_f(P) = n.$$

R is the ramification set and S the singular set ; $e_f(P) = k$ if f is conjugated to $z \mapsto z^k$ in holomorphic charts centered at P and $f(P)$.

The Riemann-Hurwitz formula

Theorem (Riemann-Hurwitz)

Let $f : X \to Y$ be a holomorphic map of degree n between compact Riemann surfaces X and Y. Then,

$$2g_X - 2 = n(2g_Y - 2) + \sum_{P \in X} (e_f(P) - 1). \tag{R-H}$$

On a proof.

Assume that Y admits a triangulation $\mathcal{T}_Y$; refining if necessary, the singular set S is contained in $\mathcal{T}_Y^{(0)}$. Lift $\mathcal{T}_Y$ to X. As f is a topological covering outside S, $|\mathcal{T}_X^{(1)}| = n|\mathcal{T}_Y^{(1)}|$ and $|\mathcal{T}_X^{(2)}| = n|\mathcal{T}_Y^{(2)}|$, while for any $Q \in S$, the number of points over Q is

$$|f^{-1}(Q)| = n - \sum [f(P) = Q](e_f(P) - 1). \qquad \square$$

On the proof of Riemann-Hurwitz formula

That Y can actually be triangulated is in full generality a nonobvious topological statement.

On the proof of Riemann-Hurwitz formula

That Y can actually be triangulated is in full generality a nonobvious topological statement. However, for Riemann surfaces Y admitting platonic tesselations (such as $\mathbb{C}\mathrm{P}^1$ with any of the Coxeter complexes seen before), this is direct, and reduces the proof of formula (R-H) to its combinatorial part.

Construction of the nonsingular metric

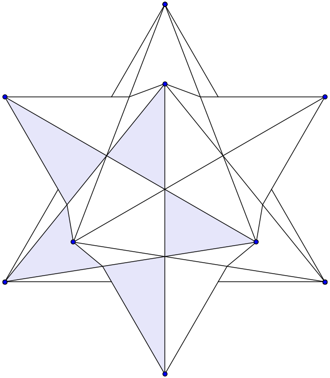

Figure: Kepler's small stellated dodecadedron (edge first)

Construction of the nonsingular metric

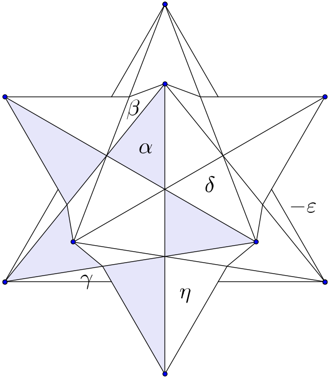

Figure: Label the faces with letters $\alpha, \ldots, \eta, -\alpha, \ldots, -\eta$

Constructing a tesselation of $\mathbb{H}^2$

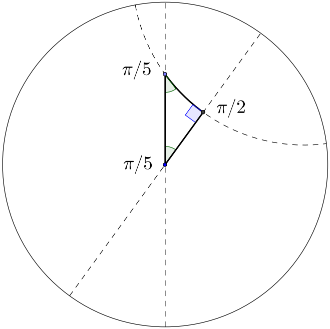

Figure: A triangle $\triangle$ with angles $\pi/2$, $pi/5$, $\pi/5$ in $\mathbb{H}^2$

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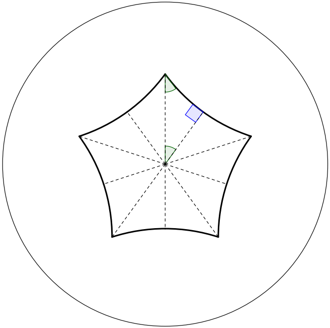

Figure: Reflect along the sides to get a hyperbolic regular pentagon into which $\{5/2\}$ can be mapped.

Unfolding $\{5/2, 5\}$ on the hyperbolic plane

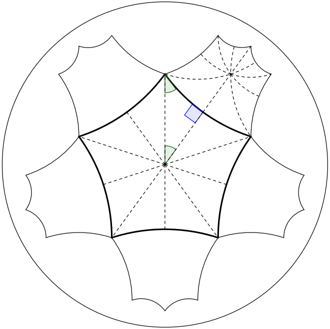

Figure: Reflect again with respect to the sides to get a tiling of $\mathbb{H}^2$ by pentagons, 5 meeting at each vertex

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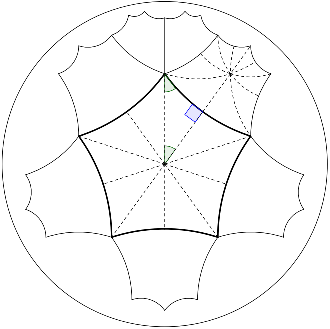

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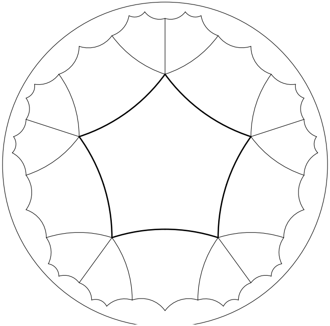

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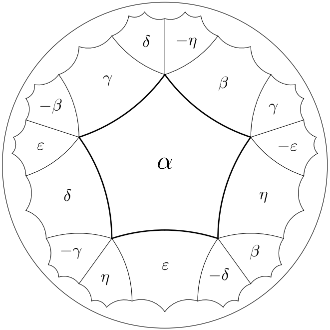

Figure: Install the hyperbolic metric on the surface.

The Riemann surface structure

The preceding construction yields a hyperbolic metric, hence a Riemann surface structure, on the small stellated dodecahedron. Denote this surface by Σ.

By construction, the mapping $\Sigma \to \mathbb{C}\mathrm{P}^1$ is holomorphic with 12 ramification points of order 2, hence the -6 Euler characteristic and that Σ has genus 4 is a consequence of Riemann-Hurwitz formula (or only of its combinatorial part).

Singular flat metric

When realized in $\mathbf{R}^3$, Σ possesses a singular flat metric with 24 singularities.

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The generalized Gauss Bonnet formula reads

$$2\pi\chi(\Sigma) = 12\pi + (-2) \cdot 12\pi.$$

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Σ also acquires a *spherical* singular metric when seen as a tiling of $\mathbb{S}^2$ with overlap.

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This can be more generaly used in order to express the degree of $\{p, q\}$: if $\mathrm{Area}^1\Sigma_{p,q}$ denotes the area of the Riemann surface of $\{p, q\}$ when given the singular spherical metric,

$$\deg\{p, q\} = \frac{\mathrm{Area}^1\Sigma_{p,q}}{4\pi} = \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{2}\right) \left|\{p, q\}^{(1)}\right|$$

Algebraic curve

Proposition (Klein)

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The associated holomorphic covering group of $\Sigma \to \mathbb{C}\mathrm{P}^1$ is (anti-naturally) isomorphic to the Galois group $\mathrm{Gal}(\mathbb{C}(z,w)/\mathbb{C}(z))$, $w^5 = z^2\frac{z+1}{z-1}$. This is a more general fact for compact Riemann surfaces, which can all be obtained as algebraic curves.

Outline

Regular convex 4-polytopes

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- Every 3-cell is realized as a geometric regular polyhedron $\{p, q\}$.
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Theorem (Schläfli)

There exists 6 convex regular real 4-polytopes, namely:

- *The simplex $\alpha_4 = \{3, 3, 3\}$.*
- *The 4-hypercube $\{4, 3, 3\}$ and its dual cross-polytope $\beta_4 = \{3, 3, 4\}$, whose vertices form short roots in C_4 and B_4.*
- *The 24-cell $\{3, 4, 3\}$ with F_4 as symmetry group.*
- *The 120-cell $\{5, 3, 3\}$ and its dual polytope $\{3, 3, 5\}$ with H_4 as symmetry group.*

Exceptional polytopes as finite quaternion groups

Identified as the unit sphere in real quaternion algebra, S^3 can be endowed with the Lie group structure of $\mathrm{SU}(2) \simeq \mathrm{Spin}(3)$, a double cover of $\mathrm{SO}(3)$.

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- Lifting the group of orientation preserving transformation of $\{3, 3\}$ yields $\{3, 4, 3\}$.
- Lifting $\mathrm{SO}(3) \cap \rho(\mathrm{W}_{3,5}) \simeq \mathfrak{A}_5$ to $G < S^3$ yields the cells of $\{5, 3, 3\}$; $\Pi = G \backslash S^3$ is Poincaré dodecahedral space. As H_4 has order 14400, G has index 120 (any doedecahedron is cut into 120 tetrahedra).

$$1 \to \{\pm 1\} \to G \to \mathfrak{A}_5 \to 1$$

$$1 \to \{\pm 1\} \to \mathrm{SU}(2) \to \mathrm{SO}(3) \to 1.$$

Conway's cuboctahedron

In the same way that polyhedra with H_3 symmetry group can be arranged on a hexagon, the 12 regular polytopes with H_4 symmetry group can be arranged on the vertices of a *cuboctahedron*.

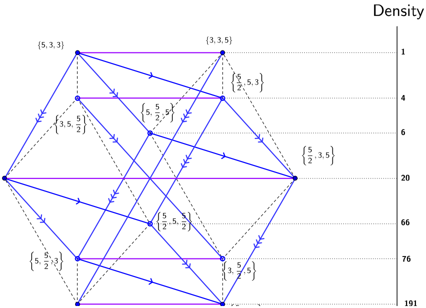

Figure: Regular polytopes with H_4 symmetry group. Arrows depict stellation, greatening and *aggrandizement* (extend 3-cells and fill).

Further reading...

References for this talk as well as complements can be found in the following.

- On star polytopes: H.S.M Coxeter, *Regular polytopes* (chapter 6).
- On Riemann surfaces and algebraic curves, their covering/Galois theory: W. Fulton, *Algebraic Topology: a first course* (chapter 20).
- On the small stellated dodecahedron and its Jacobian: M. Weber, *On Kepler's small stellated dodecahedron*, Pacific Math. journal 220 (2005), no. **1**, 167–182.
- Epistemology, around Euler's formula: I. Lakatos, *Proofs and refutations* (chapter 6).