Quasiisometries and rough isometries of solvable Lie groups

Gabriel Pallier, based on joint work with E. Le Donne and X. Xie

Heidelberg – Karlsruhe – Strasbourg Geometry day
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Karlsruher Institut für Technologie

Outline

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Quasiisometries and SOL-like groups

Quasiisometric rigidity

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Quasiisometries and SOL-like groups

Quasiisometry

Let X and Y be metric spaces.

Quasiisometry

$f : X \to Y$ is a quasiisometry if there exists $\ell, L > 0$ and $c \geqslant 0$ such that

- $\ell d(x, x') - c \leqslant d(f(x), f(x')) \leqslant L d(x, x') + c$ for all $x, x' \in X$
- $d(y, f(X)) \leqslant c$ for all $y \in Y$.

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We denote

$$\mathrm{QI}(X) = \{f : X \to X \text{ is a quasiisometry}\} / \sim$$

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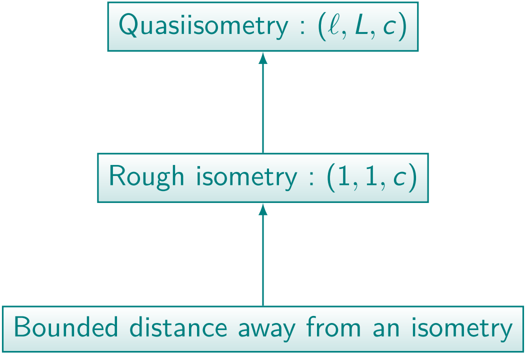
where $f \sim g$ if $\sup_x d(f(x), g(x)) < +\infty$.

- $\mathrm{QI}(X)$ is a group.
- If G is a group (finitely generated, or Lie), $\mathrm{QI}(G)$ makes sense.

Subgroups of $\mathrm{QI}(X)$



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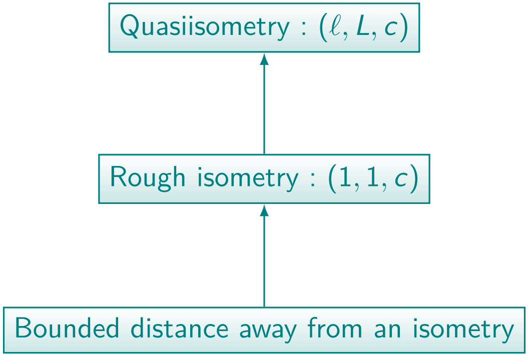

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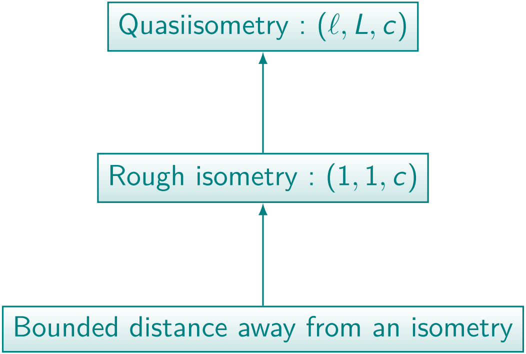

- Rough isometries of **Euclidean space** are at a bounded distance away from isometries.
- There are rough isometries of **hyperbolic space** $\mathbb{H}^n_{\mathbf{R}}$ not a bounded distance away from isometries.

The rank one solvable Lie groups

Let G be a closed subgroup of upper triangular real matrices.
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$$G = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{2t} & y \\ 0 & 0 & 1 \end{pmatrix} : t, x, y \in \mathbf{R} \right\}, N = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq \mathbf{R}^2$$

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Example : coming from simple Lie groups

$G = AN$, KAN rank one simple Lie group with trivial center.

SOL-like groups

Let G be a rank one solvable Lie group. The extension

$$1 \to N \to G \to \mathbf{R} \to 1$$

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- All the examples given before are SOL-like.
- If $\mathfrak{n}_- = 0$ or $\mathfrak{n}_+ = 0$, G is Gromov hyperbolic. The converse holds as well.

How one can think of SOL-like groups

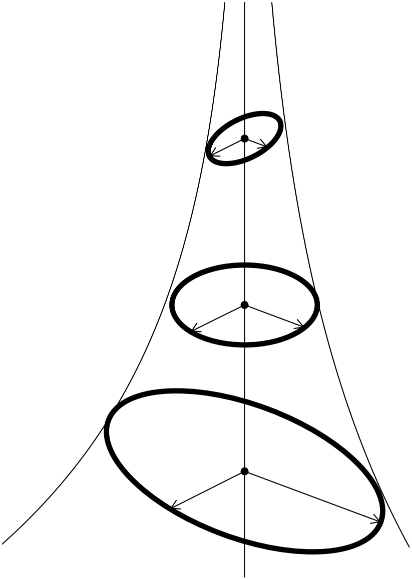
The hyperbolic ones

$$\left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{2t} & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} t \\ x \\ y \end{pmatrix} \in \mathbf{R}^3 \right\}$$

A left-invariant metric :

$$ds^2 = dt^2 + e^{-2t} dx^2 + e^{-4t} dy^2$$

"Metric view" of $\{x^2 + y^2 = \varepsilon^2\}$.



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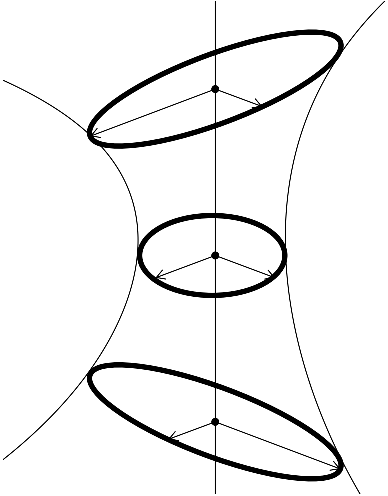
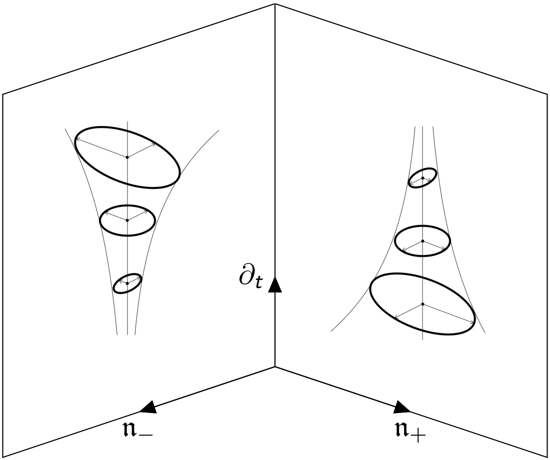


How one can think of SOL-like groups

General case

$(N_+ \times N_-) \rtimes \mathbf{R}$



Left-invariant metrics on SOL-like groups

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Theorem (Le Donne, P., Xie, 2022)

Let d_1, d_2 be left-invariant Riemannian metrics on a SOL-like group G. There is ρ (depending only on d_1 and d_2) such that $\rho d_1 - d_2$ is **bounded**.

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Corollary

Let $f \in \mathrm{QI}(G)$. If f is a rough isometry of d_1, then it is a rough isometry of d_2.

And so the subgroup $\mathrm{RI}(G) < \mathrm{QI}(G)$ of rough isometries of G makes sense for these groups.

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Let d_1, d_2 be left-invariant Riemannian distances, let ρ be such that $\rho d_1 - d_2$ is bounded.
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- f is a $(1, 1, \rho c)$ quasiisometry of ρd_1 ;
- Since d_2 and ρd_1 differ by a bounded amount, say k, f is a $(1, 1, \rho c + 2k)$ - quasiisometry of d_2.

A few words on ρ

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We can be more precise.
Let $g \notin N$. Then

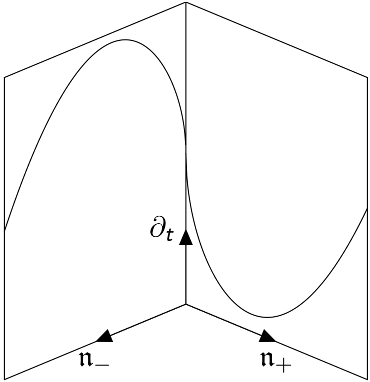
$$\rho = \frac{d_2(N, gN)}{d_1(N, gN)}.$$

Rough idea of proof

Theorem

Let d_1, d_2 be left-invariant Riemannian metrics on a SOL-like group G. There is ρ (depending only on d_1 and d_2) such that $\rho d_1 - d_2$ is **bounded**.

- During most of their lifetime, the geodesic segments of d_1 and d_2 move transversally to the cosets of N.



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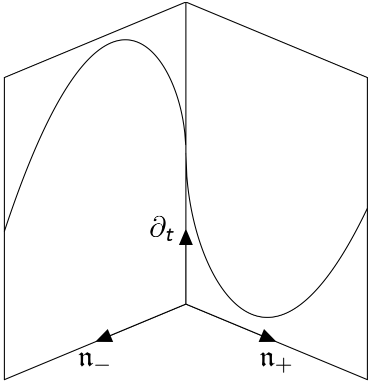

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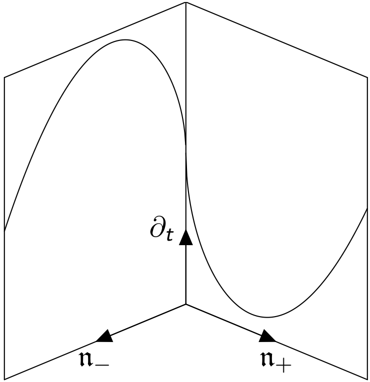
- The cosets of maximal and minimal altitude depend on $i = 1, 2$ but only up to a bounded amount.



Quasiisometric rigidity

Quasiisometries of semisimple Lie groups

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Theorem (Pansu, Kleiner-Leeb (1989 rank one, 1994 higher rank))

Let G be a semisimple Lie group, with trivial center and no factor locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. Let $X = G/K$ be the associated symmetric space. Then every quasiisometry of X is bounded distance away from an isometry of the symmetric metric on X.

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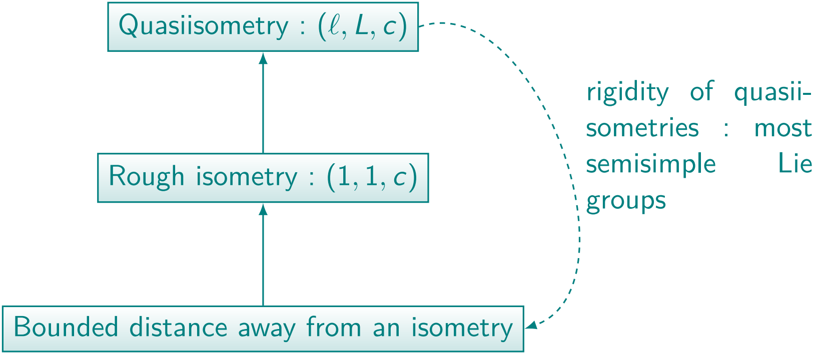

Corollary of the rigidity of quasiisometries

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QI rigidity

Let G be as before. Let Σ be a finitely generated group quasiisometric to G. Then there is a finite subgroup F of Σ such that Σ/F is a uniform lattice in G **(QI rigidity)**.

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QI rigidity holds true for uniform lattices in $G = \mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ also but the proof is different. For $G \neq \mathrm{SO}(2,1)$: a subgroup of $\mathrm{QI}(X)$ with **uniformly bounded** ℓ^{-1}, L, c can be conjugated into $\mathrm{Isom}(X)$ (Sullivan, Tukia).

QI rigidity of SOL-like groups : examples

Theorem (Eskin, Fisher, Whyte 2013) "Existential" QI rigidity

Let Σ be a finitely generated group quasiisometric to

$$\mathrm{SOL} = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} : t, x, y \in \mathbf{R} \right\}$$

Then Σ / F is a (uniform) lattice in SOL for a finite $F < \Sigma$.

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Theorem (Follows from Pansu 1989), "Inexistential" QI rigidity

There is no finitely generated group quasiisometric to

$$G = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{2t} & y \\ 0 & 0 & 1 \end{pmatrix} : t, x, y \in \mathbf{R} \right\}$$

(this group has no lattices).

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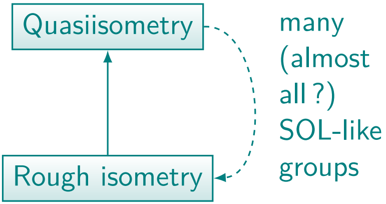
Theorem (Ferragut 22, reformulation using LDPX)

Let G be a non-unimodular, not Gromov hyperbolic SOL-type group. Then $\mathrm{QI}(G) = \mathrm{RI}(G)$.



Quasiisometric rigidity and SOL-like groups

QI rigidity conjecture for SOL-like groups

Let Σ be a finitely generated group, quasiisometric to a SOL-like G.

Σ

Then conjecturally :

1. Either $G = AN$ and Σ/F is a uniform lattice in $\widehat{G} = KAN$.
2. Or G is unimodular and Σ/F is a lattice in some $\widehat{G}$ with $G \to \widehat{G}$ co-compact.

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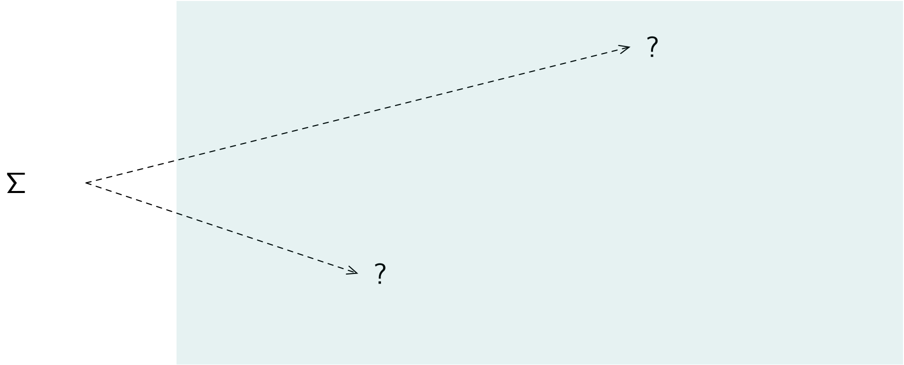

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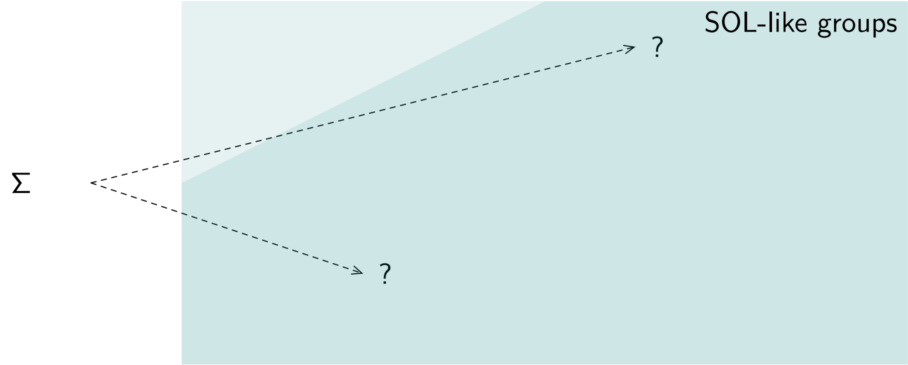

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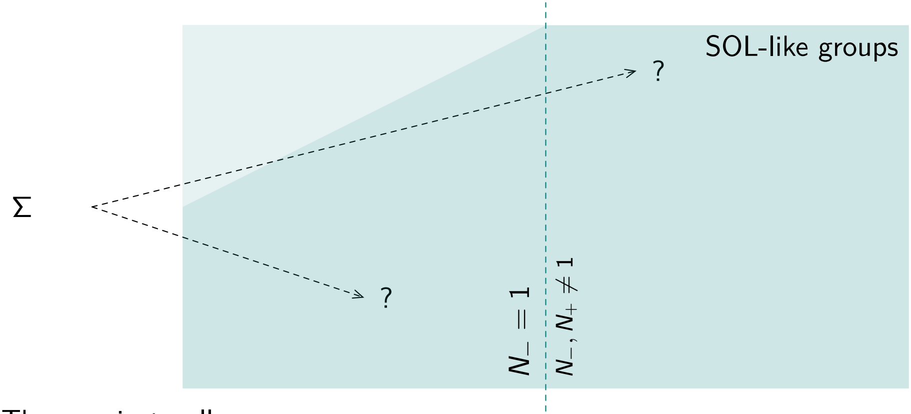

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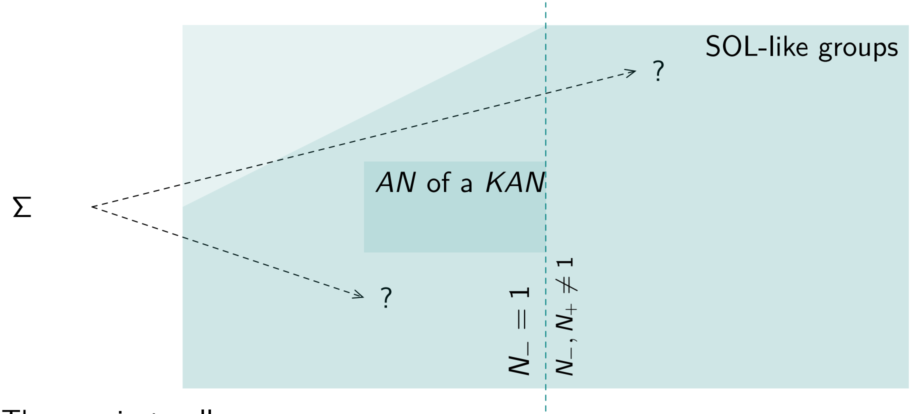

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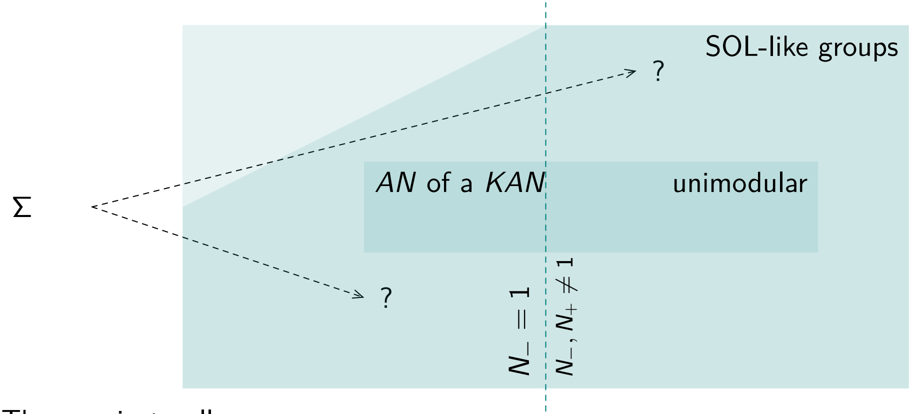

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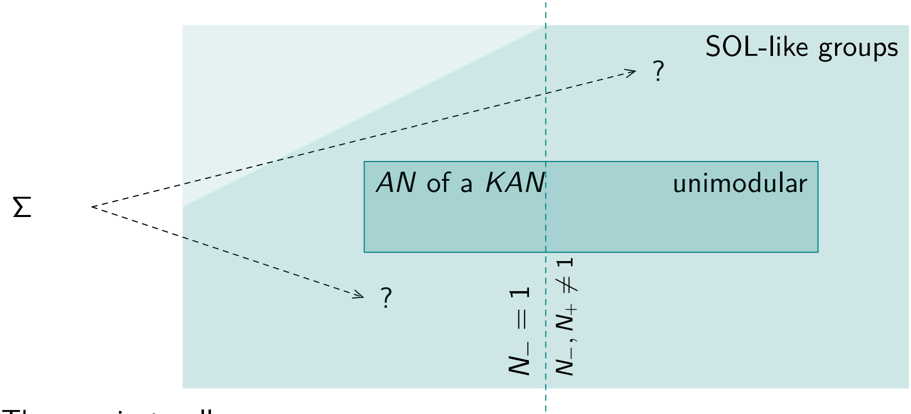

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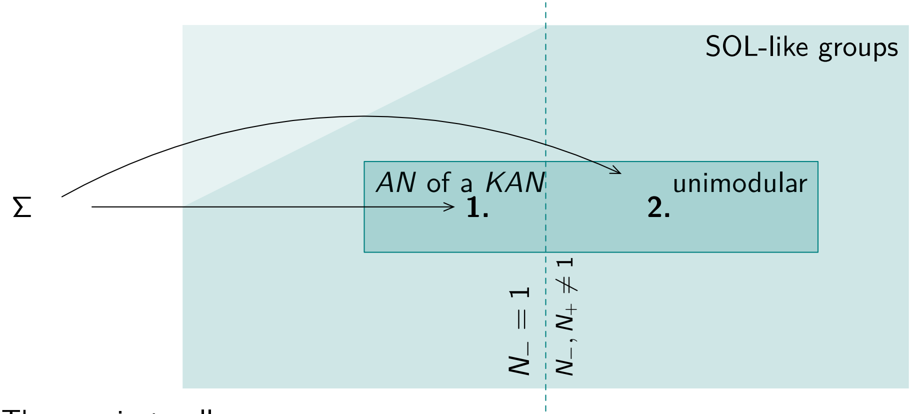

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Conjugating uniform subgroups

Carnot pairs

Let M be a nilpotent group and $D \in \mathrm{Der}(\mathfrak{m})$ such that $\mathrm{Liespan}(\ker(D-1)) = \mathfrak{m}$. We say that (M, D) is a Carnot pair.

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Theorem (Dymarz-Fisher-Xie 2023)

Let G be a SOL-like group, $(N_- \times N_+) \rtimes_{(D_-, D_+)} \mathbf{R}$ where $(N_\pm, D_\pm)$ are Carnot pairs. Let $\mathcal{Q}$ be a uniform subgroup of standard quasiisometries of G. Then $\mathcal{Q}$ can be conjugated into a standard group of standard isometries of the maximally symmetric metric of G.

Together with the description of $\mathrm{QI}(G)$, this is the key to QI rigidity for these groups.

The classification conjecture

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Theorem (Pansu 1989 Gromov-hyperbolic, Ferragut 2022 non-unimodular, Dymarz-Fisher-Xie 2023 in a greater generality)

Let G be a SOL-like group $(N_- \times N_+) \rtimes_{(D_-, D_+)} \mathbf{R}$ where $(N_\pm, D_\pm)$ are Carnot pairs. If H is a group quasiisometric to G then G and H are isomorphic.

Things I have not said : how it works

To reach a description of $\mathrm{QI}(G)$ or classify SOL-like groups up to quasiisometries one needs to do **analysis**.

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To reach a description of $\mathrm{QI}(G)$ or classify SOL-like groups up to quasiisometries one needs to do **analysis**.

- If G is a Gromov-hyperbolic SOL-like group : quasiconformal analysis on Carnot groups. A key tool is Pansu's differential. $\varphi : N \to N$ is Pansu-differentiable at $\xi \in N$ if

$$D_{\mathrm{P}}\varphi(\xi) : u \mapsto \lim_{t \to +\infty} e^{tD} \varphi(\xi)^{-1} \varphi(\xi e^{-tD} u)$$

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The main theorem that I presented today only uses “soft” methods and little analysis.

Further reading

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- *Problems on the geometry of finitely generated solvable groups*, Benson Farb & Lee Mosher (2000). ← Some problems have been solved by Eskin-Fisher-Whyte and followers, but this is still very a nice introduction with the main ideas.
- *Ingredients and consequences of quasi-isometric rigidity of lattices in certain solvable Lie groups*, Tullia Dymarz (2017 mini-course, notes can be found online).
- *Coarse differentiation of quasiisometries* (A. Eskin, D. Fisher, K. Whyte), Ann. Math 177 (2013), two papers.

Today : `http://www.pallier.org/gabriel/pdfs/gday.pdf`

Thank you !