Dehn functions and the large-scale geometry of nilpotent groups

contributed talk for GAGTA 2021

G. Pallier, joint work with C. Llosa Isenrich, R. Tessera

Nilpotent Groups

Let Γ be a finitely generated, torsion-free nilpotent group.

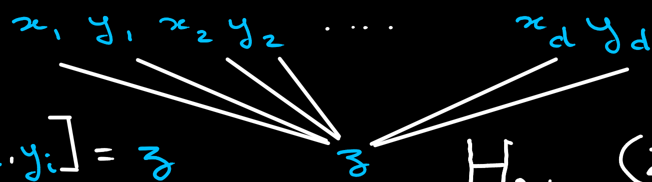
Malcev (1950s): There exists a unique simply connected Lie group G such that $\Gamma < G$ as a co-compact lattice.

We write $G = \Gamma \otimes \mathbb{R}$. Especially, Γ is finitely presented.

Let $d \geq 1$ and let R be a ring

$$H_{2d+1}(R) = \left\{ \begin{pmatrix} 1 & \xi_1 & \cdots & \xi_d & \zeta \\ & 1 & & 0 & \eta_d \\ & & \ddots & & \vdots \\ & (0) & & 1 & \eta_1 \\ & & & & 1 \end{pmatrix} \right\} < GL_{d+1}(R)$$

$H_d(\mathbb{Z}) \otimes \mathbb{R} = H_d(\mathbb{R})$



$[x_i, y_i] = z$ $\qquad H_{2d+1}(\mathbb{Z})$

HEISENBERG GROUPS

Quasi-isometries of nilpotent groups

Conjecture: Let Γ and Λ be finitely generated, torsion-free nilpotent groups. Then

$$\left[\begin{array}{c}\Gamma \text{ and } \Lambda \\ \text{are quasisometric}\end{array}\right] \iff \left[\Gamma \otimes \mathbb{R} \simeq \Lambda \otimes \mathbb{R}\right]$$

NB: 1) A finitely generated group quasi-isometric to a nilpotent group is virtually nilpotent (Gromov 1980)

2) Cannot ask "commensurable" instead of $\Gamma \otimes \mathbb{R} \simeq \Lambda \otimes \mathbb{R}$

Quasi-isometries of nilpotent groups

Let Γ and Λ be finitely generated torsion-free nilpotent.

$\boxed{\underline{\text{Th}} \text{ (Pansu 1987): If } \Gamma \text{ and } \Lambda \text{ are quasiisometric then } gr(\Gamma \otimes \mathbb{R}) \simeq gr(\Lambda \otimes \mathbb{R})}$

Where for $\underline{g}$ nilpotent Lie algebra, $\quad gr(\underline{g}) = \bigoplus_{i>0} C^i \underline{g} / C^{i+1} \underline{g}$

and $gr(G)$ has Lie algebra $gr(\text{Lie } G)$

$gr(\Gamma \otimes \mathbb{R})$ is the group structure of the asymptotic cone.

$\boxed{\underline{\text{Th}} \text{ (Shalom 2004 + Sauer 2006): If } \Gamma \text{ and } \Lambda \text{ are quasiisometric, then } H^*(\Gamma, \mathbb{R}) \simeq H^*(\Lambda, \mathbb{R}) \text{ as } \mathbb{R}\text{-algebras}}$

Dehn function

Let $\mathcal{P} = \langle S \mid R \rangle$ be a finite presentation of Γ.

$$\delta_{\mathcal{P}}(n) = \sup\left\{ \mathbf{Area}(\omega) : \omega \in \langle\langle R \rangle\rangle, \mathrm{length}_S(\omega) \leq n \right\}$$

(The asymptotics of δ only depends on $\mathcal{P}$ up to the relation $f(n) \asymp g(n)$ if $f(n) \leq Cg(Cn+C) + Cn + C$, $C \geq 1$
$g(n) \leq Cf(Cn+C) + Cn + C$

Hence $\delta_\Gamma(n) \asymp n^2, n^3, n^3 \log n \ldots$ makes sense)

$\boxed{\text{Prop: If } \Gamma \text{ and } \Lambda \text{ are quasiisometric, then } \delta_\Gamma(n) \asymp \delta_\Lambda(n).}$

Dehn function of nilpotent groups

Let Γ be finitely generated nilpotent, not virtually $\mathbb{Z}$ or 1.

- $n^2 \preccurlyeq \delta_\Gamma(n) \preccurlyeq n^{s+1}$ if $C^{s+1}\Gamma = 1$

  Gersten, Holt & Riley 2003

- If Λ is a lattice in $gr(\Gamma \otimes \mathbb{R})$ and $\delta_\Lambda(n) \preccurlyeq n^\alpha$, then: $\forall \varepsilon > 0$, $\delta_\Gamma(n) \preccurlyeq n^{\alpha+\varepsilon}$ (Papasoglu - Druțu 97)

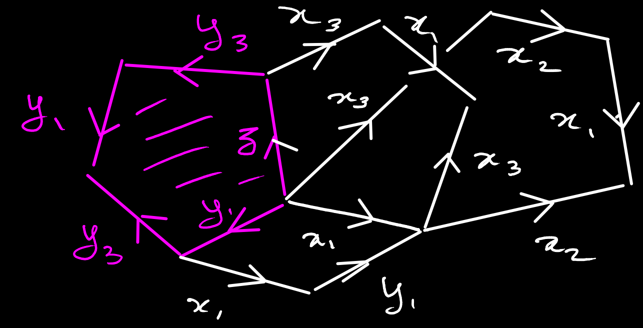
- $\delta_{H_d(\mathbb{Z})}(n) \simeq \begin{cases} n^3 & d = 3 \\ n^2 & d > 3 \end{cases}$ (Epstein et al.) (Gromov, Allcock, Olshanskii - Sapir)

  more precise later

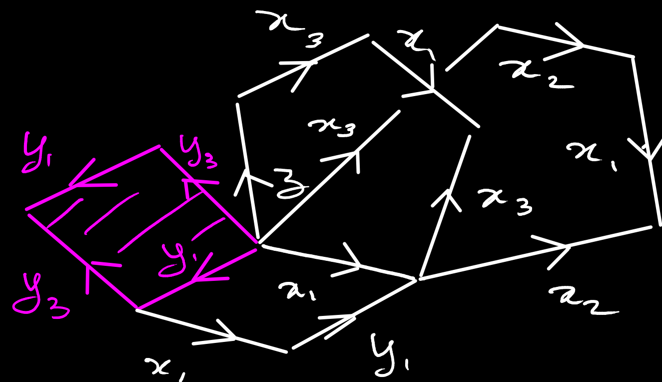
$\boxed{\begin{array}{l}\underline{\text{Th}} \text{ (Llosa Isenrich, P., Tessera 2020): There are pairs } \{\Gamma, \Lambda\} \text{ of} \\ \text{finitely generated, torsion-free nilpotent groups such that} \\ gr(\Gamma \otimes \mathbb{R}) \simeq gr(\Lambda \otimes \mathbb{R}) \ \underline{\text{and}}\ \delta_\Gamma(n) \not\simeq \delta_\Lambda(n)\end{array}}$

The Groups (lowest-dimensional pair)

$$\Gamma = \left\langle x_1, x_2, y_1, y_3 \;\middle|\; \begin{array}{ll} [x_1,x_2]=x_3 & [x_i,y_j]=1 \\ [x_1,x_3]=z & [x_i,z]=1 \\ [y_1,y_3]=z & [y_j,z]=1 \end{array} \right\rangle$$



$$\Lambda = \left\langle x_1, x_2, y_1, y_3 \;\middle|\; \begin{array}{ll} [x_1,x_2]=x_3 & [x_i,y_j]=1 \\ [x_1,x_3]=z & [x_i,z]=1 \\ [y_1,y_3]=1 & [y_j,z]=1 \end{array} \right\rangle$$



$\mathrm{gr}(\Gamma \otimes \mathbb{R}) \simeq \Lambda \otimes \mathbb{R}$ $\qquad$ $\delta_\Lambda(n) \asymp n^4$ (e.g. by Gersten-Holt-Riley)

$$\boxed{\underline{\text{Th}}\ (\text{LPT}):\ \delta_\Gamma(n) \asymp n^3}$$

lower bounds : cohomology

upper bound : algorithm

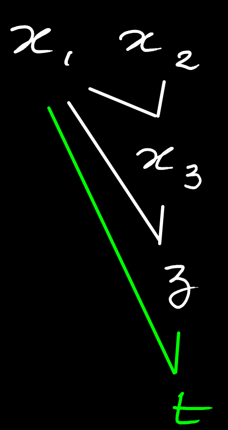
Lower bounds

Let Γ be a finitely presented group.

If $1 \to \underset{\langle t \rangle}{\overset{\shortparallel}{\mathbb{Z}}} \xrightarrow{\text{central}} \tilde{\Gamma} \to \Gamma \to 1$ and $\langle t \rangle$ has $\underbrace{\text{distortion } \Delta(n)}_{\inf\{m \,:\, t^m \in B_{\tilde{\Gamma}}(1,n)\}}$ in $\tilde{\Gamma}$,

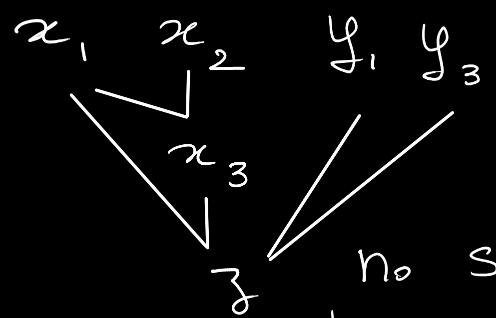
then $\delta_\Gamma(n) \geq \Delta(n)$

<u>Obs</u> (Cornulier 2016) The centralized Dehn function (largest distortion of a central extension) differs for Γ and Λ



$\delta_\Lambda(n) \succcurlyeq n^4$

n^4 – distorted (Osin)



loss of this central extension

no such lower bound for δ_Γ

$b_2(\Lambda) = 7$

$b_2(\Gamma) = 6$

(Some idea of) Upper bound $\delta_\Gamma(n) \lesssim n^3$

Let $w \in \langle\langle R_\Gamma \rangle\rangle \cap F_{\{x_1, x_2\}}$ of length at most n.

A) Put w in normal form with Area $O(n^3)$

$$w = [x_1^{m_k}, x_3^{n_k}][x_1^{\ell_k}, x_3] \dots [x_1^{m_1}, x_3^{n_1}][x_1^{\ell_1}, x_3]$$

where $k \leq n$, $|m_i n_i + \ell_i| \leq n^2$ $|m_i|, |n_i| \leq 3\sqrt{|m_i n_i + \ell_i|}$

B) If $[x_1^{m_k}, x_3^{n_k}][x_1^{\ell_k}, x_3] \dots [x_1^{m_1}, x_3^{n_1}][x_1^{\ell_1}, x_3] \in \langle\langle R_\Gamma \rangle\rangle$ with k, m_i, n_i, ℓ_i as above, then this word has Area $\leq Ckn^2$

Step A) uses the extra relation $[x_1, x_3] = [y_1, y_3]$ to make subwords of the form $[x_1^{m_i}, \quad]$ travel at low cost.

Geometrically: the extra relator is used at intermediate scale.

Other results & perspectives

We can also prove:

- There are nilpotent groups with centralized Dehn $\not\asymp$ Dehn (known by Wenger 2011; gap improved)
- Γ and Λ are non-quasiisometric in a strong, quantitative form (Sublinear Bilipschitz Equivalence).

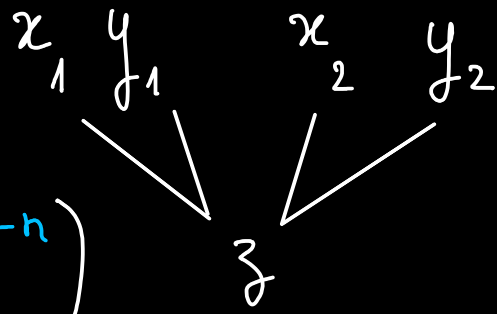
Questions:

- Can one adapt the technique to more nilpotent groups?
- What is the rescaled limit of the combinatorial fillings of $[x_1^n, [x_1^n, [x_1^n, x_2^n]]]$ as $n \to \infty$ in the asymptotic cone $\Lambda \otimes \mathbb{R}$?

Thank you!

« Make subwords travel at low cost »

$$\Gamma = H_5(\mathbb{Z}), \quad n = k$$



$$
\begin{aligned}
x_1^n y_1^n &= x_1^{n-k} x_1^{k} y_1^{k} y_1^{n-k} \\
&= x_1^{n-k} y_1^{k} x_1^{k} z^{k^2} y_1^{n-k} \\
&= x_1^{n-k} y_1^{k} x_1^{k} [x_2^k, y_2^k] \, y_1^{n-k} \\
&= x_1^{n-k} y_1^{k} x_1^{k} y_1^{n-k} [x_2^k, y_2^k] \\
&= \cdots \\
&= y_1^n x_1^n z^{n^2}
\end{aligned}
$$

$$\mathrm{Area}\left([x_1^k, y_1^k] z^{-n}\right)$$

$$\mathrm{Area} = O(kn) = O(k^3)$$

Bootstrap $\rightsquigarrow \mathrm{Area}_{H_5}\left([x_1^n, y_1^n] z^{-n^2}\right) = O\left(n^{2+\varepsilon}\right)$

$\forall \varepsilon > 0$

Area of a word

Def: Let $\mathcal{P}=\langle S \mid R\rangle$ be a finite presentation of Γ.

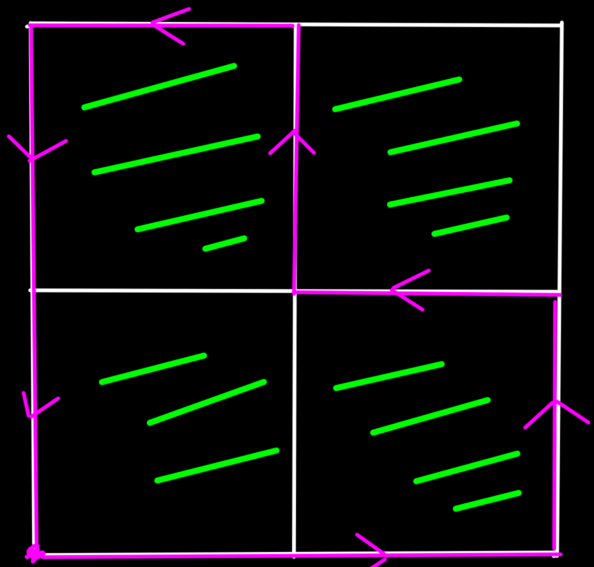
Let $w\in\langle\langle R\rangle\rangle$. Define $\mathrm{Area}_{\mathcal{P}}(w) := \inf\left\{k\geq 0 : w \underset{free}{=} \prod_{j=1}^{k} g_j r_j^{\pm 1} g_j^{-1}\right\}$

Example: $\mathcal{P}=\langle x,y \mid xyx^{-1}y^{-1}\rangle$ $\quad\Gamma=\mathbb{Z}^2$

$$x^2y^2x^{-2}y^{-2} \underset{free}{=} x^2y^2x^{-1}yx^{-1}y^{-2} \cdot (y^2xy^{-1})\, xyx^{-1}y^{-1}\, (y^2xy^{-1})$$

mod $\langle\langle R\rangle\rangle$

$$= x^2y^2x^{-1}yx^{-1}y^{-2} = \dots = 1.$$



Universal covering of the presentation complex

$$\boxed{\mathrm{Area}_{\mathcal{P}}(x^2y^2x^{-2}y^{-2}) = 4}$$

The Riemannian Dehn function

<u>Th</u> (Bridson, Burillo–Taback):

Let the finitely presented Γ act geometrically on the simply connected Riemannian manifold X.

$$\delta_\Gamma(n) \simeq \sup_{\substack{\gamma : S^1 \to X \\ \text{rectifiable loop of length } \leq n}} \inf \left\{ \mathrm{Area}(\Delta) : \begin{array}{l} \Delta \text{ is a Lipschitz} \\ \text{filling } B^2 \to X \\ \text{with } \partial\Delta = \gamma \end{array} \right\}$$