Dehn functions and the large-scale geometry of nilpotent groups

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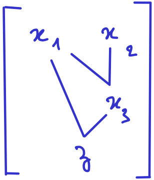
Let Γ and Λ be infinite, torsion-free, nilpotent, finitely generated groups.

Think about $\mathbb{Z}^2$, $H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$

3 - nilpotent group

$\Gamma = \langle x_1, x_2, x_3, z \mid [x_1, x_2] = x_3,\ [x_1, x_3] = z,\ [x_i, z] = 1 \text{ for all } i \rangle$



Question

If Γ and Λ are quasiisometric, what can be said about Γ and Λ?

Quasiisometry: Make $\Gamma \curvearrowright X$ properly cocompactly on geodesic metric
$\Lambda \curvearrowright Y$

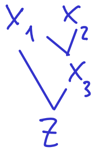
a quasiisometry is a map $\phi : X \to Y$ such that there are constant L, C

- $-C + \frac{1}{L} d(x,y) \leq d(\phi(x), \phi(y)) \leq L\, d(x,y) + C$ for every $x, y \in X$
- $\forall z \in Y$, $d(z, \phi(X)) \leq C$

May think of X as a universal cover of a Riemannian compact manifold

Malcev 1951 : there exists a simply connected nilpotent Lie group G $=$"$\Gamma \otimes \mathbf{R}$" such that $\Gamma \hookrightarrow G$ with finite kernel and image a uniform lattice.

- $\mathbb{Z}^2 \qquad G = \mathbb{R}^2$
- $H_3(\mathbb{Z})$, $\qquad G = H_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$
- 3-nilpotent filiform group $\qquad$ G a Lie group with Lie algebra



Nomizu 1954 : if $G = \Gamma \otimes \mathbf{R}$ then $H^*(\Gamma, \mathbf{R}) = H^*(G, \mathbf{R})$

- Betti numbers $\quad b_k(\Gamma) \leq \binom{\mathrm{vcd}(\Gamma)}{k}$ and if there is equality everywhere then Γ is abelian

Conjecture (folklore) : Γ and Λ are quasiisometric $\iff$ $\Gamma \otimes \mathbb{R}$ and $\Lambda \otimes \mathbb{R}$ are isomorphic.

direct

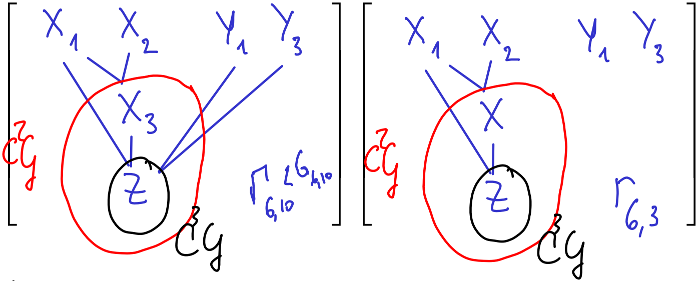
Let $\mathcal{G}$ be the nilpotent Lie algebra of a group G. Define its contraction

$$\mathcal{G}_\infty = \bigoplus_{i>0} C^i\mathcal{G}/C^{i+1}\mathcal{G}$$

"Simpler" Lie algebra

$(\mathcal{H}_3)_\infty = \mathcal{H}_3$ ← Heisenberg algebra is called Carnot

with the induced Lie brackets and denote G_∞ the associated Lie group.



1. $(G_\infty)_\infty = G_\infty$; we call G Carnot if $G = G_\infty$
2. Contracting preserves the nilpotency class.
3. G_∞ is "more abelian" than G, for instance $b_p(G_\infty) \geqslant b_p(G)$ for all p.

$b_1(G)=4$ $b_1(G_\infty)=4$ in our example, but $b_2(G)=6$ $b_2(G_\infty)=7$

Contraction : $\mathcal{G}_\infty = \bigoplus_{i>0} C^i\mathcal{G}/C^{i+1}\mathcal{G}$

Theorem (Pansu, 1980s) If Γ and Λ are quasiisometric, then

$$(\Gamma \otimes \mathbf{R})_\infty \simeq (\Lambda \otimes \mathbf{R})_\infty.$$

This uses asymptotic cones. Look at Γ and Λ "from infinity"

You see $(\Gamma \otimes \mathbb{R})_\infty$ and $(\Lambda \otimes \mathbb{R})_\infty$ with subRiemannian metrics.

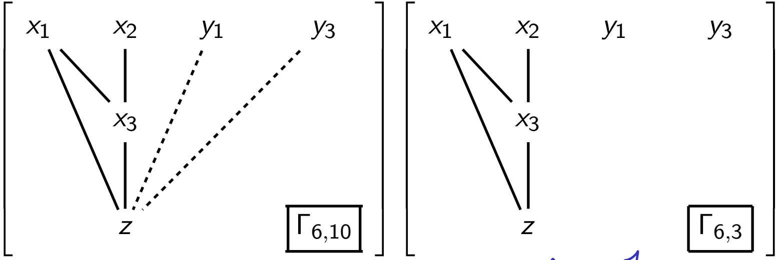
Uses Geometry and some analysis.

Theorem (Shalom, 2004) If Γ and Λ are quasiisometric, then for all p

$$b_p(\Gamma) = b_p(\Lambda).$$

Relies on reformulating the QI into a Uniform Measure Equivalence.

A geometric method to tell some groups with same asymptotic cones apart



Same asymptotic cones

$\Gamma_{6,3} \otimes \mathbb{R} = (\Gamma_{6,10} \otimes \mathbb{R})_{\oplus 0}$

Theorem (Llosa Isenrich, Pallier, Tessera, 2020)

The Dehn function of $\Gamma_{6,10}$ grows like n^3 while the Dehn function of $\Gamma_{6,3}$ grows like n^4. ← the Dehn function is a quasiisometry invariant.

The Dehn function

$\Gamma = \pi_1(M) \leftarrow M$ nilmanifold

(a) For the geometric group theorist Let $\langle S \mid \mathcal{R} \rangle$ be a finite presentation of Γ, w a (freely reduced) word over S

$\mathrm{Area}(w) = |w|_{\langle\langle \mathcal{R} \rangle\rangle}$ length as a product of conjugates of relators

$\delta_\Gamma(n) = \sup_{\ell(w) \leqslant n} \mathrm{Area}(w)$.

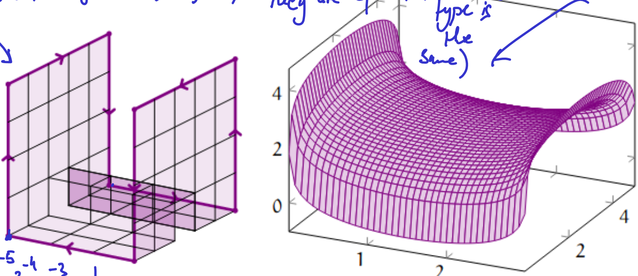
(b) For the differential geometer Let M be a nilmanifold, $\Gamma = \pi_1(M)$, $\gamma : S^1 \to M$ Lipschitz.

$\mathrm{Area}(\gamma) = \inf \int_{\Delta^2} |\Lambda^2 d\phi|$ taken over $W^{1,n}$ maps $\phi \colon \Delta^2 \to M$ with trace γ. $\delta_\Gamma(r) = \sup_{\ell(\gamma) \leqslant r} \mathrm{Area}(\gamma)$.

$\Gamma = \mathbb{Z}^3 \quad \langle x, y, z \mid [x,y], [y,z], [z,x] \rangle$ They are equivalent (the growth type is the same)

Drawing in $\tilde{M}$

Universal cover of the presentation complex



$z^4 y^5 z^{-4} x^3 z^4 y^{-5} z^{-4} x^{-3} = 1$

(a) (b)

Some general upper bounds

Easy Lemma $\delta_{G\times H}(n) \preccurlyeq \max\{n^2, \delta_G(n) \times \Delta_H(n)\}$.
Theorem 1 (Gromov 1994) If G is a Carnot group (that is, $G = G_\infty$) of class c then $\delta_G(n) \preccurlyeq n^{c+1}$.
Theorem 2 (Papasoglu 1996) For every $\alpha > 1$, if $\delta_{(\Gamma\otimes\mathbf{R})_\infty}(n) \preccurlyeq n^\alpha$, then $\delta_\Gamma(n) \preccurlyeq n^{\alpha+\varepsilon}$ for all $\varepsilon > 0$. *Proof in R. Young's "Notes on asymptotic cones".*

For $\Gamma_{6,10}$ and $\Gamma_{6,3}$:

$\delta_{\Gamma_{6,3}}(n) \preccurlyeq n^4 = n^{3+1}$ by Gromov

$\delta_{\Gamma_{6,10}}(n) \preccurlyeq n^{4+\varepsilon}$ by Papasoglu

(Actually, $\delta_{\Gamma_{6,10}}(n) \preccurlyeq n^4$ by Gersten-Holt-Riley 2003)

A lower bound

(a) Geometric group theorist Let

$$1 \to \mathbf{Z} \to \overline{\Gamma} \to \Gamma \to 1$$

be a **central** extension of Γ, with 1 sent to $s \in \overline{\Gamma}$. If $w \in F_{\overline{S}}$ represents s^n, then

$$\mathrm{Area}(w) \geqslant \ell_{s \cup \overline{S}}(s^n).$$

Highly distorted central extensions.

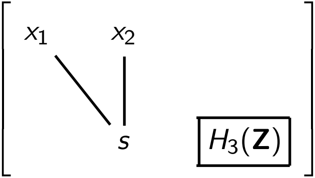
↳ Dehn function that grows fast

(b) Differential geometer Let β be a left-invariant 2-form on $\Gamma \otimes \mathbf{R}$ with closed a primitive α. ~~$\overline{S}$ lift of S in $\overline{\Gamma}$.~~

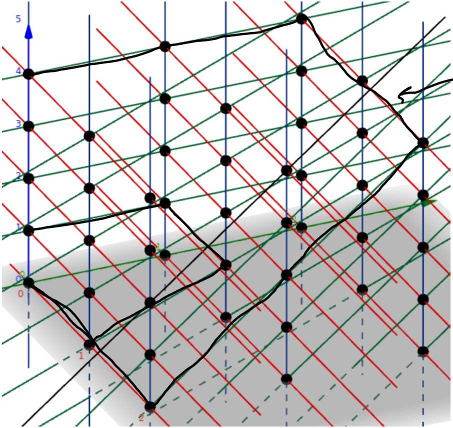
$$\mathrm{Area}(\gamma) \geqslant C \left| \int_\gamma \alpha \right|$$

2-forms with "heavy" primitives.

Queen Dido and the Heisenberg group



The isoperimetric inequality in $\mathbb{R}^2$ is quadratic



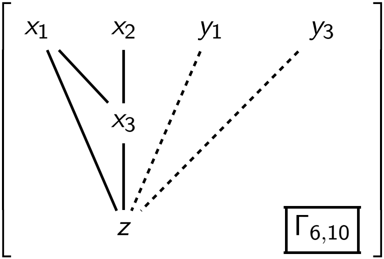
$[x_1^n, y_1^n] = s^{n^2}$

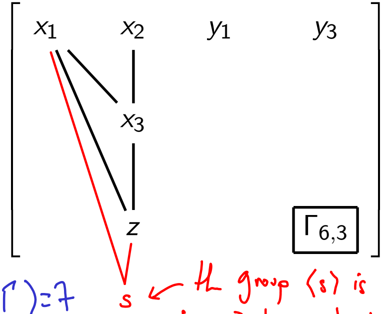
$\langle s \rangle$ is quadratically distorted and geodesics to it project to optimal isoperimetric loops in the plane

$\mathbb{R}^2$, X, Y dx, dy dual basis

$\beta = dx \wedge dy$ $\alpha = x\,dy$

Back to $\Gamma_{6,10}$ and $\Gamma_{6,3}$





$b_2(\Gamma_{6,10}) = 6$

no n^4-distorted central extension

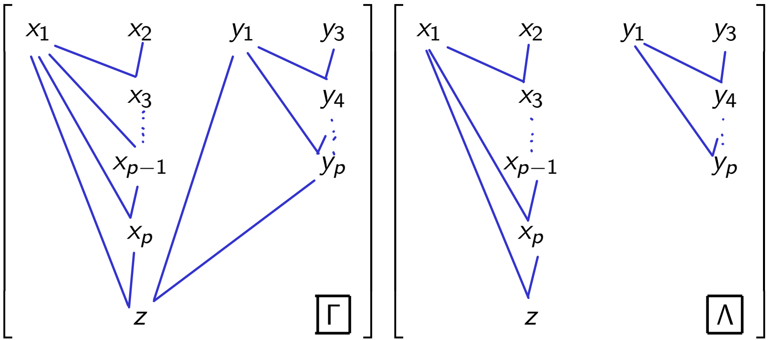
$b_2(\Gamma_{6,3}) = 7$

the group $\langle s \rangle$ is n^4-distorted in a 7 dim central extension $\tilde{\Gamma} \to \Gamma_{6,3}$

So $\delta_{\Gamma_{6,3}}(n) \succcurlyeq n^4$, and $\delta_{\Gamma_{6,3}}(n) \simeq n^4$

Lower bound given by cohomology is only $n^3 \preccurlyeq \delta_{\Gamma_{6,10}}(n)$

Central products of filiform groups



Theorem (Llosa Isenrich, Pallier, Tessera, 2020)

The Dehn function of Γ grows like n^p while the Dehn function of Λ grows like n^{p+1}.

If p is odd, $b_2(\Lambda) - b_2(\Gamma) = 1$. ← This is why the lower bound given by cohomology differs

If p is even, $b_2(\Lambda) - b_2(\Gamma) = 2$. ! ← The lower bound given by cohomology is n^{p-1}

Using forms with bounded differentials

Gersten
Block - Weinberger

Bull. Soc. math. France,
98, 1970, p. 81 à 116.

**COHOMOLOGIE DES ALGÈBRES DE LIE NILPOTENTES.
APPLICATION A L'ÉTUDE
DE LA VARIÉTÉ DES ALGÈBRES DE LIE NILPOTENTES**

PAR

MICHÈLE VERGNE.

Introduction.

A la suite de M. GERSTENHABER, plusieurs auteurs ont publié des articles consacrés à l'étude de la variété des structures algébriques d'un certain type (structure d'algèbres associatives, d'algèbres de Lie, etc.) portées par un espace vectoriel fixe V. Il est apparu qu'il y avait un

Vergne 1970 In even dimension $\geqslant 6$, there are two Carnot filiform algebras. In odd dimension, there is only the "standard one".

1. If p is odd, there is a n^p-distorted central extension that provides the lower bound
2. If p is even, this extension fails to exist. The most distorted central extension is n^{p-1}-distorted

$(\xi_1, \dots \xi_p, \zeta, \eta_1, \eta_3, \dots, \eta_p)$ dual basis to $(X_1, \dots, X_p, Z, \dots, Y_p)$

Vergne cocycle: $\xi_2 \wedge \xi_p + \xi_3 \wedge \xi_{p-1} + \dots$

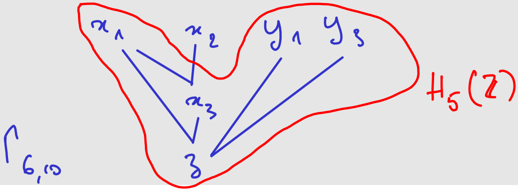
$\rightsquigarrow$ So instead of an invariant form β we use a bounded form β

It gives the lower bound n^p for δ via the differential geometry approach.

A few words on the upper bound

Theorem [LI-P-T]

$\delta_{\Gamma_{6,10}}(n) \preccurlyeq n^3$.



Start : **Gromov, Turston, Allcock, Olshanskii-Sapir :** $\delta_{H_5(\mathbf{Z})} \asymp n^2$.

announcement

proofs

In $H_5(\mathbb{Z})$, horizontal loops have horizontal fillings

Cannot draw $H_5(\mathbb{Z})$

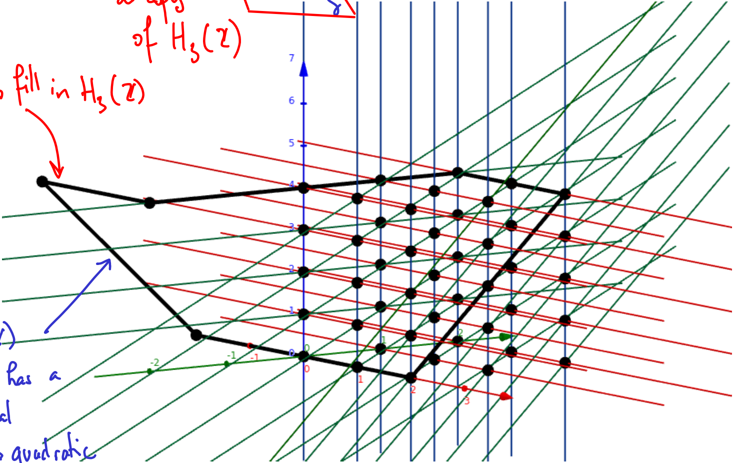
$x_1 \quad x_2 \quad y_1 \quad y_2$

z

a copy of $H_3(\mathbb{Z})$

this loop is hard to fill in $H_3(\mathbb{Z})$

the filling is cubic.

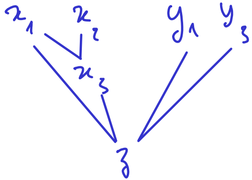


in $H_5(\mathbb{Z})$ this loop has a horizontal filling $\rightsquigarrow$ quadratic

Lemma 1 (Changing factors)

Every word w in x_1, x_3 representing a central element in $\Gamma_{6,10}$ is homotopic (rewritable) to the same word over y_1, y_3 with cost $O(\ell(w)^2)$.

Pf: x_1, x_3, y_1 and y_3 are in a copy of $H_5(\mathbb{Z})$. So this already follows from $\delta_{H_5}(n) \lesssim n^2$ (Olshanskii-Sapir)



representing the central element.

Lemma 2

In order to prove the upper bound one needs only an algorithm do reduce ~~nul homotopic~~ words of length n in x_1, x_2 at cost $O(n^3)$.

Pf: At first have a word over x_1, x_2, y_1, y_3.
We can separate into two subwords over x_1, x_2 and y_1, y_3
Then, replace the word over y_1, y_3 with a word in x_1, x_3 by Lemma 1
Finally $x_3 = [x_1, x_2]$.

To a product of rectangle words

There is an algorithm, in part inspired from Olshanskii and Sapir proof of $\delta_{H_3}(n) \simeq n^2$, to reduce words

For $m, n, \ell \geqslant 0$ define $T = T(m, n, \ell) = [x_1^m, x_3^n][x_1^\ell, x_3]$.
(It represents $z^{mn+\ell}$.)
Let $w(x_1, x_2)$ be a word of length $\leqslant n$.
We repeat $O(n)$ times the following process at cost $O(n^2)$ each time.

- Move all the instances of x_1 to the left starting with the left-most.
- After moving an x_1 to the left, move all x_3s created in the process to the left.
- Move all the $T(m, n, 0)$ words created in the process to the right.

After repeating this i times the word has the form

$$x_3^{k_1} x_1^{k_2} x_2^{k_3} x_1^{\pm 1} v(x_1, x_2) \prod_{j \leqslant i} T_{i-j}$$

with $|k_2| + |k_3| + 1 + \ell(v) \leqslant n$ and $k_1 \leqslant in$.

Claim

Each application of the 3 items above needs a cost $O(n^2)$.

We end with a product of a power of and rectangle words, $\prod T_{l-i}$, $l \leqslant n$.

Claim (Similar to Olshanskii-Sapir's Lemma)

Let $l > 0$ and let $T_i = T(m_i, n_i, l_i)$, $1 \leqslant i \leqslant l$ be words with $|m_i \cdot n_i + l_i| \leqslant n^2$ and $|m_i|, |n_i| \leqslant 3n$. Assume that $\prod_{i=1}^{l} T_i$ is null-homotopic. There is a constant $C_2 > 0$ such that the identity

$$\prod_{i=1}^{l} T_{l-i} \equiv 1$$

holds in $\Gamma_{6,10}$ with area $\leqslant C_2 \cdot l \cdot n^2$.

Thank you for your attention.