Ergodic Maps and the cohomology of nilpotent Lie groups

The aim of today is to show proposition 1.2.
First, let us recall the statement of theorem 1.1:

Theorem. Let G and H be two simply connected nilpotent Lie groups and let $\psi \in Y_0$ be an ergodic map.
Then, ψ induces a morphism of cohomology algebras:

$$H^*(\mathfrak{h};\mathbb{R}) \longrightarrow H^*(\mathfrak{g};\mathbb{R})$$
$$[\omega] \longmapsto [\bar{\psi}^*\omega]$$

being $\bar{\psi}^*\omega$ the amenable average of ω under ψ. Furthermore, if ψ is a quasi-isometry, then the induced map is an isomorphism.

As we can see, the problem with the above result is the fact that the existence of an ergodic map is assumed.
Proposition 1.2 deals with this problem:

Proposition. Let $f \in Y_0$, then there is a sequence $(g_n)_{n\in\mathbb{N}} \subset G$ s.t the sequence $(f.g_n)_{n\in\mathbb{N}}$ converges uniformly on compact sets to a ergodic map ψ. Moreover, if f is a quasi-isometry, then so is ψ.

Hence, if we have a quasi-isometry $f \in Y_0$, the above prop. guarantees the existence of an ergodic quasi-isometry ψ.

Before showing proposition 1.2, let us recall some definitions and to establish some notation. Let us fix two simply connected Lie groups G and H.

Then,

$$C^{\infty,\infty}(G) = \{ f \in C^\infty(G) : Df \in L^\infty(G) \text{ for all } D \in \mathcal{U}(\mathfrak{g}) \}$$

Here, $\mathcal{U}(\mathfrak{g})$ is the space of left-inv. differential operators.

Suppose

$$\mathfrak{g} = \langle X_1, \dots, X_n \rangle$$
$$\mathfrak{h} = \langle Y_1, \dots, Y_m \rangle$$

Given a smooth map $\alpha : G \to H$, if we consider its differential at $p \in G$, $d\alpha_p : T_p G \to T_{\alpha(p)} H$, there are smooth functions $m_{ij} : G \to \mathbb{R}$, with $1 \le i \le m$, $1 \le j \le n$ s.t

$$d\alpha_p (X_i(p)) = \sum_{K=1}^{m} m_{K,i}(p)\, Y(\alpha(p))$$

With this in mind, we define

$$Y_0(G,H) = \{ \alpha \in C^\infty(G,H) : \alpha(e_G) = e_H \text{ and } \{m_{ij}\}_{i=1,j=1}^{m\ \ n} \subset C^{\infty,\infty}(G) \}$$

We can define an action on Y_0 as follows: if $\alpha \in Y_0$ and $g \in G$, we define

$$(\alpha . g)(x) = \alpha(g)^{-1}\, \alpha(gx)$$

Exercise. Check that $\alpha . g \in Y_0$.

Remark:

(1) If $\alpha \in Y_0$ is a fix point, then

$$\alpha . g = \alpha \quad \forall \ g \in G$$

$$\Rightarrow \alpha(g)^{-1}\, \alpha(gx) = \alpha(x) \quad \forall\, g \in G$$

$$\Rightarrow \alpha(gx) = \alpha(g)\alpha(x)$$

i.e., α is a Lie group homomorphism.

Def. A smooth map $f: G \to H$ is said to be ergodic if there is a G-invariant ergodic prob. measure μ on Y_0 s.t for any continuous function $A : Y_0 \to \mathbb{R}$ we have

$$\lim_{n \to \infty} \int_{B_n} A(f \cdot g)\, dg = \int_{Y_0} A \, d\mu \qquad (0)$$

Remark: μ being a G-invariant ergodic prob. measure means

1. For any measurable set $E \subset Y_0$, $\mu(E . g) = \mu(E)$ for all $g \in G$.
2. If E is a G-invariant set, i.e., $E . g = E \ \forall\, g \in G$ then $\mu(E) = 0$ or $\mu(E) = 1$.

Examples:

1. $G = H = \mathbb{R}$ and define $f(t) := t + \sin(t)$. Then, clearly $f \in Y_0$.

Set $K = \overline{f.G}$, by definition:

$$(f.t)(x) = f(t+x) - f(x)$$
$$= \underbrace{\sin(t+x) + x - \sin(t)}_{h_t}$$

Notice that $h_t = h_{t+2\pi}$, i.e., the function depends entirely on t modulo 2π. Thus,

$$K = \{ h_t : t \in [0, 2\pi] \}$$

Now, remember that if ϕ is a 2π-periodic function, then

$$\lim_{n \to \infty} \frac{1}{2n} \int_{-n}^{n} \phi(g)\, dg = \frac{1}{2\pi} \int_{0}^{2\pi} \phi(t)\, dt$$

Notice that, given a fixed $A \in C(Y_0)$, the function defined by

$$\phi(t) := A(h_t)$$

is 2π-periodic.

Therefore,

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} A(F\cdot g)\, dg$$

$$= \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} \phi(g)\, dg$$

$$= \frac{1}{2\pi} \int_0^{2\pi} A(h_t)\, dt \qquad (1)$$

Let ν be the normalized Lebesgue measure on $[0, 2\pi]$.
Consider the map

$$\Phi : \mathbb{R}/2\pi\mathbb{Z} \longrightarrow Y_0$$
$$t \longmapsto h_t$$

If we consider the pushforward measure of ν w.r.to Φ, we have:

$$\int_{Y_0} A\, d\Phi_*(\nu) = \frac{1}{2\pi} \int_0^{2\pi} A(h_t)\, dt$$

Now, the limit measure μ is the <u>unique</u> Radon measure satisfying eq. (0). Consequently, $\mu = d\Phi_*(\nu)$.

2. Now, let us see an example of prop. 1.2.

Take again $G = H = \mathbb{R}$ and this time

$$\psi(x) = x - 1 + \sqrt{x^2 + 1}$$

It can be checked that $\psi \in Y_0$. We have that

$$\begin{aligned}(\psi \cdot g)(x) &= \psi(g+x) - \psi(g) \\ &= x + \sqrt{(g+x)^2 + 1} - \sqrt{g^2+1}\end{aligned}$$

If $g \to +\infty$, we have that $(\psi \cdot g)(x) \longrightarrow 2x$.
Similarly, if $g \to -\infty$, $(\psi \cdot g)(x) \longrightarrow 0$. That being said, let $h_+(x) = 2x$ and $h_-(x) = 0$.
Therefore, in this case

$$\overline{\psi . G} = \{\psi \cdot g : g \in \mathbb{R}\} \cup \{h_+, h_-\}$$

Next,

$$\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} A(\psi . g)\, dg = \lim_{n\to\infty} \left(\frac{1}{2n} \int_{-n}^{0} A(\psi . g)\, dg \right.$$

$$\left. + \frac{1}{2n} \int_{0}^{2n} A(\psi . g)\, dg \right) = \frac{1}{2} A(h_-) + \frac{1}{2} A(h_+)$$

Thus, the limit measure is

$$\mu = \frac{1}{2}\delta_{h_-} + \frac{1}{2}\delta_{h_+}$$

Clearly, $\mu(\{h_+\}) = \frac{1}{2}$. Nonetheless:

$$\begin{aligned}(h_+ \cdot g)(x) &= h_+(g+x) - h_+(g) \\ &= 2(g+x) - 2g \\ &= 2x \\ &= h_+(x)\end{aligned}$$

i.e., the singleton $\{h_+\}$ is G-invariant but its measure is not zero. Thus, μ is not ergodic and thus Ψ is not an ergodic map. Nonetheless, it follows that the sequences

$$(\Psi \cdot n)_{n \in \mathbb{N}}$$
$$(\Psi \cdot (-n))_{n \in \mathbb{N}}$$

converge to ergodic maps, namely, h_+ and h_-.

We now proceed with the proof of prop. 1.2, which makes us use of the following lemmas:

Lemma. Let $f \in Y_0$. Then, there is a G-invariant ergodic prob. measure μ supported on $K = \overline{f \cdot G}$.

Theorem (Lindenstrauss) Let G be an amenable grp equipped with a left inv. Haar measure and let (X, η) be a prob. space equipped with an ergodic left-action of G.
Suppose $(F_n) \subset G$ is a tempered Følner sequence. Then for any $f \in L^1(\eta)$ the following holds for η-a.e. $x \in X$:

$$\lim_{n \to \infty} \int_{F_n} f(g \cdot x)\, dg = \int_X f\, d\eta$$

Remarks: One can show $(B_n) \subset G$ form a tempered Følner sequence.

- A tempered Følner sequence is a Følner seq. (F_n) s.t $\exists\ C>0$ s.t

$$\left| \bigcup_{K \le n} F_K^{-1} F_{n+1} \right| \le C |F_{n+1}|$$

Proof of prop. 1.2:

Let $K = \overline{f.G}$ be the compact orbit closure of f. By the above lemma there is a G-invariant ergodic prob. measure μ supported on K.

We know that Y_0 is metrizable and therefore, since $K \subset Y_0$, we can see K as a compact metric space.

Consequently, $C(K)$ is separable and thus, it admits a countable dense subset $S = \{s_i\}_{i=1}^{\infty}$.

For each $s_i \in S$, Lindenstrauss' theorem guarantees that there is a measurable space $Z_n \subset K$ s.t $\mu(Z_n) = 0$ and

$$\lim_{n\to\infty} \int_{B_n} s_i(y.g) = \int_K s_i \, d\mu \qquad (1)$$

for all $K - Z_n$. Set $E_n = K - Z_n$ and define

$$E := \bigcap_{n=1}^{\infty} E_n$$

We have that $\mu(E) = 1$ and thus $E \neq \emptyset$.

Claim. Given $\phi \in E$, the following equality holds for all $\varphi \in C(K)$:

$$\lim_{n\to\infty} \fint_{B_n} \varphi(\phi \cdot g)\, dg = \int_K \varphi \, d\mu$$

Indeed, one has that

$$\left| \fint_{B_n} \varphi(\phi\cdot g)\, dg - \int_K \varphi \, d\mu \right| \leq \left| \fint_{B_n} (\varphi(\phi\cdot g) - S_i(\phi\cdot g))\, dg \right|$$

$$+ \left| \fint_{B_n} S_i(\phi\cdot g)\, dg - \int_K S_i \, d\mu \right| + \left| \int_K (S_i - \varphi)\, d\mu \right|$$

The first and third terms can be made as small as desired due to the fact that $\overline{S} = C(K)$. By eq. (1) the second term goes to zero as $n \to \infty$. This shows that ϕ is an ergodic map.

As $\phi \in K = \overline{F.G}$, there is a sequence $(F.g_n)_{n\in\mathbb{N}} \subset F.G$ s.t $F.g_n \to \phi$ uniformly on compact sets. Set $\phi_n := F.g_n$.

Now suppose that f is an (L, C)-quasi-isometry, i.e., we have:

(1) $\frac{1}{L}\, d(x,y) - C \leq d(f(x), f(y)) \leq L\, d(x,y) + C$

(2) $\forall\, y \in H,\ \exists\, x \in G$ s.t $d(y, f(x)) \leq C$.

Note that

$$d(\phi_n(x), \phi_n(y)) = d\left(f(g_n)^{-1} f(g_n x),\ f(g_n)^{-1} f(g_n y)\right)$$
$$= d(f(g_n x), f(g_n y))$$
$$\leq L\, d(g_n x, g_n y) + C$$
$$= L\, d(x, y) + C$$

Similarly, $\frac{1}{L} d(x,y) - C \leq d(\phi_n(x), \phi_n(y))$.

Taking limits, it follows that

$$\frac{1}{L} d(x,y) - C \leq d(\phi(x), \phi(y)) \leq L\, d(x,y) + C \quad \forall\, x, y \in G$$

Now, let $y \in H$ be fixed. Consider, for each $n \in \mathbb{N}$ the element $f(g_n) y \in H$; by eq. (2) $\exists\ x_n \in G$ s.t

$$d(f(x_n), f(g_n) y) \leq C$$

$$d\left(f(g_n)^{-1} f(x_n), y\right) \leq C$$

Now,

$$f(g_n)^{-1} f(x_n) = f(g_n)^{-1} f\big(g_n \underbrace{(g_n^{-1} x)}_{z_n}\big)$$
$$= \phi_n(z_n)$$

Thus, we get

$$d(\phi_n(z_n), y) \leq C \qquad (3)$$

Now, notice that

$$d(z_n, e_G) \leq L\left(d(\phi_n(z_n), e_H) + C\right)$$

$$\leq L\left(d(\phi_n(z_n), y) + d(y, e_H) + C\right)$$

$$\leq L\left(C + d(y, e_H) + C\right)$$

i.e., $(z_n)_{n\in\mathbb{N}}$ is bounded and thus it admits a convergent subsequence $(z_{n_k}) \subset (z_n)$.

Let $x^* \in G$ be the limit of this subsequence. Next, we have:

$$d(\phi_{n_k}(x_{n_k}), \phi(x^*)) \leq d(\phi_{n_k}(x_{n_k}), \phi(x_k)) + d(\phi(x_{n_k}), \phi(x^*))$$

As $k \to \infty$, the 1st term vanishes by uniform convergence and the second vanishes by continuity of ϕ. Thus,

$$\lim_{n\to\infty} \phi_{n_k}(x_{n_k}) = \phi(x^*)$$

Using (3), it follows that

$$d(\phi_{n_k}(x_{n_k}), y) \leq C$$

Thus,

$$d(\phi(x^*), y) \leq C$$

i.e., ϕ is a quasi-isometry.

□