On Shalom's paper

"Harmonic analysis, cohomology and the large scale geometry of amenable groups"

Main thm: Γ, Λ two fin gen nilp gps

$\Gamma \overset{QI}{\sim} \Lambda \Rightarrow b_k(\Gamma) = b_k(\Lambda) \ \forall k \in \mathbb{N}$

$\dim_{(2)} H^k(\Gamma, \mathbb{R})$

Strategy: 1) Relate QI between amenable gps to the notion of measure equivalence (ME).

2) Relate cohom of two ME gps: inductions of unitary reps through a ME coupling

3) For nilp gps, invoke $H^k_{(2)}$ to conclude.

I - ME and QI

Def: Two countable gps Γ and Λ are measure equivalent if $\exists (\Omega, m)$ std Borel space on which $\Gamma \curvearrowright \Omega \curvearrowleft \Lambda$ measure preserving and s.t.:
- the actions commute
- they admit finite measure Borel fund domains X_Γ, X_Λ

$\Omega = \bigsqcup_{\lambda \in \Lambda} \gamma X_\Gamma = \bigsqcup_{\lambda \in \Lambda} \lambda X_\Lambda$. $m(X_\Gamma) = m(X_\Lambda) < \infty$.

examples: 1) Γ, Λ two lattices in the same lcsc gp G.

$\Omega = (G, m_G) \quad \Gamma \curvearrowright G \curvearrowleft \Lambda$

2) $\Lambda \leq \Gamma$ finite index subgp.

Rem: (Shapiro's lemma) $\Lambda \leq \Gamma$ fin index subgp (π, V) Λ-module

$H^k(\Lambda, \pi) \cong H^k(\Gamma, \underbrace{\operatorname{Ind}_\Lambda^\Gamma \pi}_{\text{induced module}})$

$L^2(\Gamma/\Lambda, V)$

3) (Gaboriau-Weiss) Any two infinite cbles amenable gps are OE.

↳ refine the notion of ME. (by ME)

Def: Two gps Γ and Λ are unimodular ME if

- $\forall \gamma \in \Gamma \ \exists S_\gamma \subset \Lambda$ finite $\quad \gamma X_\Lambda \subseteq S_\gamma X_\Lambda = \bigsqcup_{\lambda \in S_\gamma} \lambda X_\Lambda$.
- $\forall \lambda \in \Lambda, \ \exists S_\lambda \subset \Gamma$ finite $\quad \lambda X_\Gamma \subseteq \bigsqcup_{\gamma \in S_\lambda} \gamma X_\Gamma$

Thm 2.1.7: If two (fin gen) amenable Γ and Λ are QI then for some finite (abelian) gp M, $\exists$ an ergodic UME (Ω, m) between Λ and $\Gamma \times M$, $X_{\Gamma \times M} \subseteq X_\Lambda$, ($\Gamma \times M \curvearrowright \Lambda$-action on (Ω, m) is regular)

$\left(QI \Rightarrow \text{"}L^\infty\text{-ME + "mutually cobounded"} \right)$

First step: Construction of a "topological coupling"

Fix $\phi_0 : \Lambda \to \Gamma$ a (c_0, D)- QI $\quad c_0 \geq 1, D \geq 0$.

$\exists Q \subseteq \Lambda$ finite; $\phi_0(\lambda) = \phi_0(\lambda') \Rightarrow \lambda^{-1}\lambda' \in Q$ $(D_{\Gamma}(1, c_0 D) = Q)$

Pick any finite gp M with $|M| \geq |Q|$, we make $\phi_0 : \Lambda \to \Gamma \times M$ injective, $(c_0, 0)$-QI (same c_0, D)

Define: $\Omega = \{ \phi : \Lambda \to \Gamma \times M$ injective $F(c_0 D)$-QI $\}$

- Γ acts by precomposition, free since $\phi \in \Omega$ are inj
- $(\Gamma \times M)$ acts by post-compositions, free

} actions are by isometries, thus commute.

$X_{\Gamma \times M} = \{ \phi \in \Omega, \ \phi(1_\Lambda) = 1_{\Gamma \times M} \}$.

↳ is a fund dom for $(\Gamma \times M)$-actions

↳ compact (Arzelà-Ascoli).

Let $E_{\gamma^{-1}} = \{ \phi \in \Omega \quad \phi(1_\Gamma) = \gamma^{-1} \} \quad \gamma \in \Gamma \times M$

$K_\gamma = \Lambda E_{\gamma^{-1}} = \{ \phi \in \Omega$ attaining γ somewhere$\}$.

Claim: $\Gamma \times M = \{ \gamma_0^{\pm 1}, \gamma_1, \gamma_2, \dots \}$

$X_\Lambda = E_{1_{\Gamma \times M}} \cup \bigcup_{i=0}^{n} (E_{\gamma_i} \cap K_{\gamma_1 \gamma_i} \cap \dots \cap K_{\gamma_0^{\pm 1}})$

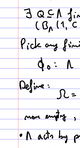
$X_{\Gamma \times M}$

is a fund domain for Λ-actions, and compact.

$X_{\Gamma \times M} \subset X_\Lambda$.

Second step: top $\Rightarrow$ meas.

1. Cocycles and actions on fund dom.



$c_\Lambda : \Gamma \times X_\Lambda \to \Lambda$

$c_\Lambda(\gamma, x) \ \gamma x \in X_\Lambda$

$\gamma \curvearrowright x$

$\Gamma \curvearrowright X_\Lambda$

$\Gamma \times M$

$\Gamma \curvearrowright \Omega \curvearrowleft \Lambda$

$c_\Gamma : \Lambda \times X_\Gamma \to \Gamma$

$c_\Gamma(\lambda, y) \ \lambda y \in X_\Gamma$.

$\lambda \curvearrowright y$

$\Lambda \curvearrowright X_\Gamma$.

2. Use of amenability.

X_Γ is compact $\Rightarrow$ $\mathrm{Prob}(X_\Gamma)$ is compact for weak* top

$\Lambda \curvearrowright X_\Gamma$ now $\Lambda \curvearrowright \mathrm{Prob}(X_\Gamma)$ compact in a loc convex top vect sp.

Λ amenable $\Rightarrow$ $\exists \underset{\text{finite}}{\nu} \in \mathrm{Prob}(X_\Gamma)$ Λ-inv.

Kaimanovich-Vershik $\Rightarrow \exists \mu \in \mathrm{Prob}(X_\Gamma)$ Λ-inv$^{\text{ergodic}}$.

3. $\Omega = \bigsqcup_{\delta \in \Gamma} \delta X_\Gamma$ extend μ Γ-invariantly. Obtain a measure on Ω Γ-inv, Λ-inv for $\Lambda \curvearrowright \Omega \curvearrowleft \Lambda$ (since it is with Γ-inv $\Gamma \curvearrowright \Omega$, Λ-inv $\Lambda \curvearrowright X_\Gamma$)

4. Uniformity $\quad \lambda X_\Gamma \underset{\text{compact}}{\subset} \bigsqcup_{\delta \in \Gamma} \underset{\text{open compact}}{\delta X_\Gamma} \Rightarrow$ extend μ finitely. □

II - Inductions of reps and their cohom

Λ, Γ ME. $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ unitrep of Λ.

$\rightsquigarrow \mathrm{Ind}_\Lambda^\Gamma \pi : \Gamma \to \mathcal{U}(L^2(X_\Lambda, \mathcal{H}))$ induced rep.

$\underset{L^2(X_\Lambda, \mathcal{H})}{(\mathrm{Ind}_\Lambda^\Gamma \xi)(\gamma)} = \pi(c_\Lambda(\gamma^{-1}, x)^{-1}) F(\gamma^{-1} x) = \underset{\Gamma \curvearrowright X_\Lambda}{\sigma_\Lambda(\gamma, x)} \xi$

Def: Given a cocycle $c \in C^k(\Omega, \pi) := C(\Lambda^{k+1}, \mathcal{H})^\Lambda$ we define the induced cocycle $\mathrm{Ic} : X_\Lambda \to \mathbb{R}$

$\mathrm{Ic}(\gamma_0, \dots, \gamma_k)(x) = c(\sigma_\Lambda(\gamma_0, x), \dots, \sigma_\Lambda(\gamma_k, x))$.

Q: Does $\mathrm{Ic}(\gamma_0, \dots, \gamma_k) \in L^2(X_\Lambda, \mathcal{H})$?

Prop: If Γ and Λ are UME then $\mathrm{Ic}(\gamma_0, \dots, \gamma_k) \in L^2(X_\Lambda, \mathcal{H})$ $\forall c \in C^k(\Lambda, \pi) \quad \forall \gamma_0, \dots, \gamma_k \in \Gamma$

[Also $\mathrm{Ic} \in C^k(\Gamma, \mathrm{Ind}\pi) = C(\Gamma^{k+1}, L^2(X_\Lambda, \mathcal{H}))^\Gamma$

hence $\mathrm{I} : C^k(\Lambda, \pi) \to C^k(\Gamma, \mathrm{Ind}\pi)$

[I is cont and commutes with coboundary

$\Rightarrow \mathrm{I} : H^k(\Lambda, \pi) \to H^k(\Gamma, \mathrm{Ind}\pi)$

$\overline{H}^k(\Lambda, \pi) \to \overline{H}^k(\Gamma, \mathrm{Ind}\pi)$.

Proof: $\forall \gamma \in \Gamma$, $S_\gamma \subset \Lambda$ finite $\quad \gamma X_\Lambda \subset \bigsqcup_{\lambda \in S_\gamma} \lambda X_\Lambda$

assume that $c_\Lambda(\gamma, X_\Lambda) \subset S_\gamma^{-1}$.

$\mathrm{Ic}(\gamma_0, \dots, \gamma_k) \in L^2$ since it only takes finitely many values.

$\| \mathrm{Ic}(\gamma_0, \dots, \gamma_k) \|_{L^2}^2 = \int_{X_\Lambda} \| c(\sigma_\Lambda(\gamma_0, x), \dots, \sigma_\Lambda(\gamma_k, x)) \|^2 \, dm(x)$

$= \sum_{\substack{\text{finite} \\ < \infty}} \dots \cdot \underbrace{m(X_\Gamma)}_{< \infty} \leq m(X_\Lambda) < \infty$.

Thm: If Λ and Γ are UME with $X_\Gamma \subseteq X_\Lambda$ then the maps $\mathrm{I} : H^k(\Lambda, \pi) \to H^k(\Gamma, \mathrm{Ind}\pi)$

$\overline{H}^k(\Lambda, \pi) \to \overline{H}^k(\Gamma, \mathrm{Ind}\pi)$ are injective.

Proof: Transfer operators: $\mathrm{T} : C^k(\Gamma, \mathrm{Ind}\pi) \to C^k(\Lambda, \pi)$.

s.t. T is cont, commutes with d.

$\mathrm{T} \circ \mathrm{I} = m(X_\Gamma) \cdot \mathrm{id}$.

$\mathrm{T}\beta(\lambda_0, \dots, \lambda_k) = \int_{X_\Gamma} \underbrace{\beta(\sigma_\Gamma(\lambda_0, x), \dots, \sigma_\Gamma(\lambda_k, x))}_{L^2(X_\Lambda, \mathcal{H})}(x) \, dm(x)$

Well def:

$= \sum_{\substack{\gamma_0, \dots, \gamma_k \in \Gamma \\ \text{finitely many}}} \int_{A_{\gamma_0 \dots \gamma_k}} \beta(\gamma_0, \dots, \gamma_k)(x) \, dm(x)$

$A_{\gamma_0 \dots \gamma_k}$ partition X_Γ.

$(\mathrm{T} \circ \mathrm{I})(c)(\lambda_0, \dots, \lambda_k)$

$= \int_{X_\Gamma} c(\underbrace{\sigma_\Gamma(\lambda_0, x)}_{= \lambda_0}, x), \dots, \underbrace{\sigma_\Gamma(\lambda_k, x)}_{= \lambda_k}, x)) \, dm(x)$

$= m(X_\Gamma) c(\lambda_0, \dots, \lambda_k)$ since $X_\Gamma \subseteq X_\Lambda$.

□

III - Proof of Main Thm

Γ, Λ fin gen nilp gps, QI

Part I $\Rightarrow$ M fin abelian gp, $(\Omega, m, X_\Lambda, X_{\Gamma \times M})$ ergodic UME coupling with $X_{\Gamma \times M} \subseteq X_\Lambda$.

π = trivial rep of Λ on $\mathbb{R}$ $\rightsquigarrow \mathrm{I}\pi$ = Koopman rep of $\Gamma \times M$ on $L^2(X_\Lambda)$ (acts with integral $= 0$ $\quad \Gamma \times M \curvearrowright X_\Lambda$

We have $L^2(X_\Lambda) = L^2_0(X_\Lambda) \oplus \mathbb{R}$

$\Gamma \times M \curvearrowright X_\Lambda$ ergodic $\Rightarrow$ $L^2_0(X_\Lambda)$ has no inv vectors.

$H^k(\Lambda, \mathbb{R}) \overset{\text{Part II}}{\hookrightarrow} H^k(\Gamma \times M, L^2(X_\Lambda))$

$H^k(\Lambda, \mathbb{R})$ fin dim $= H^k(\Gamma \times M, L^2_0(X_\Lambda)) \oplus H^k(\Gamma \times M, \mathbb{R})$

$= 0$ since $\Gamma \times M$ is nilp so it has $H^{>0}_b$ (Eilenberg)

$H^k(\Gamma \times M, \mathbb{R})$

$H^k(\Lambda, \mathbb{R}) \hookrightarrow H^k(\Gamma \times M, \mathbb{R}) \underset{\text{classical Simplex}}{\cong} H^k(\Gamma, \mathbb{R})$

$\Rightarrow b_k(\Lambda) \leq b_k(\Gamma)$

Reverse roles $b_k(\Lambda) = b_k(\Gamma)$.

□