Quasi-isometries and cohomologies of nilpotent Lie groups

after Antonelli and Young

Consult the version v3 from July 18th on arXiv.

(I) Quasiisometries and nilpotent groups

G, H simply connected Lie groups, equipped with left-invariant Riemannian metrics d_G, d_H.

$\phi : G \to H$ is a quasiisometry if $\exists \lambda \geq 1, \exists c \geq 0$,

- $\forall x, x' \in G$, $\frac{1}{\lambda} d(x,x') - c \leq d(f(x), f(x')) \leq \lambda d(x,x') + c$
  (f is a q.i. embedding)
- $\forall y \in H$, $d(y, \phi(G)) \leq c$.

NB: This definition makes sense for metric spaces.

Conj (folklore): G, H csc nilpotent Lie groups.
G and H are quasiisometric $\iff$ G and H are isomorphic (as Lie groups)

Why nilpotent Lie groups?

Th (Malcev 49 + Gromov 81 + Gordon-Wilson 86 + ... Losert 2020)
G compactly generated locally compact group, of polynomial growth, i.e $\mathrm{Haar}(B_G(1,r)) = O(r^N)$ for some N.
Then $\exists\, W(G) \triangleleft G$ maximal compact
$\exists\, G_{nil}$ csc nilpotent Lie group,

$$G/W(G) \xrightarrow[\text{co-compact}]{\text{closed,}} G_{nil} \rtimes Q, \quad Q \text{ compact}.$$

Quasi-isometrically "harmless".

QI - classify csc nilpotent Lie groups $\Rightarrow$ QI - classify all groups of polynomial growth.

Examples: • $G = \mathbb{Z}^d$: $W(G) = 1$, $G_{nil} = \mathbb{R}^d$

• G finitely generated nilpotent: $W(G) = \mathrm{Tor}(G)$, $Q = 1$.

$G/\mathrm{Tor}(G) \xrightarrow[\text{lattice}]{\text{uniform}} G_{nil}$ (Malcev)

Moreover, $\dim H_i(G,\mathbb{R}) =: b_i(G) = b_i(\mathfrak{g}_{nil}) := \dim H^i(\mathfrak{g}_{nil}, \mathbb{R})$

Nomizu 1954

Why quasiisometry and isomorphism?

Prop: G, H csc nilpotent Lie. TFAE
(i) G and H are isomorphic as Lie groups
(ii) $\exists M$ smooth manifold, $G \curvearrowright M \curvearrowleft H$ smooth commuting, free transitive actions

Prop (Gromov): Same assumption on G and H. TFAE:
(i) G and H are quasiisometric
(ii) $\exists X$ locally compact Hausdorff, $G \curvearrowright X \curvearrowleft H$ continuous commuting, proper co-compact actions

Some words on proofs:
(i) $\Rightarrow$ (ii) is easy on the left and actually the hardest part on the right
(ii) $\Rightarrow$ (i) is an "exercise" on the left which helps to handle (ii) $\Rightarrow$ (i) on the right.

On the left: (i) $\Rightarrow$ (ii) $G \curvearrowright G \curvearrowleft G$ by left and right multiplication
(ii) $\Rightarrow$ (i) Fix $x_0 \in M$. For $g \in G$, there is a unique $h \in H$ such that $g \cdot x_0 = x_0 \cdot h^{-1}$. Set $\phi(g) := h$

Then, for every $(g_1, g_2) \in G^2$ one has

$$\begin{aligned} g_1 g_2 \cdot x_0 = g_1 \cdot g_2 \cdot x_0 &= g_1 \cdot x_0 \cdot \phi(g_2) \\ \| \qquad\qquad\qquad &= x_0 \cdot \phi(g_1) \cdot \phi(g_2) \\ x_0 \cdot \phi(g_1 g_2) \qquad &= x_0 \cdot \phi(g_1)\phi(g_2) \end{aligned}$$

By freeness $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$

On the right: (ii) $\Rightarrow$ (i) works the same way, but now set $\phi(g)$ such that $g \cdot x_0$ is near $x_0 \cdot \phi(g)$

By construction ϕ will be "nearly surjective" and a "coarse embedding", namely subsets of bounded diameter go within subsets of uniformly bounded diameter in H.

(i) $\Rightarrow$ (ii) is by far the hardest.

After Gromov, cheat by assuming $\phi : G \to H$ is a Lipschitz q-i.

Pick $X_0 = C^0(G, H)$ with the compact-open topology

$\cup$

X = closure of $G \cdot f_0 \cdot H$ in X_0.

$G \curvearrowright X \curvearrowleft H$ is continuous, proper and co-compact.

(II) Cohomological invariants.

Th (Shalom 04): Let G, H be csc nilpotent groups with lattices Γ, Λ (Malcev) which have bases with structure cst in $\mathbb{Q}$.

If G and H are q.i. then $b_i(\Gamma) = b_i(\Lambda)$ for all $i \geq 0$

One important ingredient in the proof of this theorem is what is now called "Shalom's property H_T^n": for G csc nilpotent Lie group

for every unitary representation π of G, $\overline{H}^m(G,\pi) \neq 0$ implies that π (strongly) contains the trivial representation.

(NB: Here $\overline{H}$ means we take $\frac{\ker d_m}{\operatorname{im} d_{m-1}}$)

This is essentially Blanc's theorem from the 1970s (see Marc's talk)

Th (Sauer 06): Same assumptions on G and H.
Then $H^*(\mathfrak{g},\mathbb{R}) \simeq H^*(\mathfrak{h},\mathbb{R})$ as $\mathbb{R}$-algebras with cup product.

Rk: This is really finer, consider for instance

$$b_i^j := \operatorname{rank}\left(H^i(\mathfrak{g},\mathbb{R}) \otimes H^j(\mathfrak{g},\mathbb{R}) \longrightarrow H^{i+j}(\mathfrak{g},\mathbb{R})\right)$$
$$\alpha \otimes \beta \longmapsto \alpha \wedge \beta$$

By Sauer's theorem this is a QI-invariant.

Also if $\dim \mathfrak{g} = 4k$, then the signature (Poincaré duality)

of $H^{2k}(\mathfrak{g},\mathbb{R}) \otimes H^{2k}(\mathfrak{g},\mathbb{R}) \to H^{4k}(\mathfrak{g},\mathbb{R}) \simeq \mathbb{R}$
$\alpha \otimes \beta \longmapsto \alpha \wedge \beta$ is a QI-invariant.

Th (Gotfredsen - Kyed 20 using Bader 18)
Same conclusion as Sauer, without assuming that G and H have lattices.

But the isomorphism of cohomology algebras is not very explicit.

Theorem 1.1 (Antorelli - Young)

Let G and H be connected, simply connected nilpotent Lie groups. Let $\psi : G \to H$ be an ergodic map. Let $d \geq 0$. Let $\omega \in \Lambda^d(\mathfrak{h}, \mathbb{R})$ be a left-invariant differential d-form, and let $\lambda \in \Lambda_d(\mathfrak{g}, \mathbb{R})$ be a left-invariant d-vector field.

Then $\overline{\psi^*}\,\omega(\lambda) := \lim_{n \to +\infty} \fint_{B_G(1,n)} \psi^* \omega(\lambda)\, dg$ exists,

and ψ^* induces a homomorphism $\overline{\psi}^* : H^*(\mathfrak{h}, \mathbb{R}) \to H^*(\mathfrak{g}, \mathbb{R})$

If ψ is a q.i. then $\overline{\psi}^*$ is an isomorphism